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Physics Mech.

SOLUTIONS

OF THE

EXAMPLES

APPENDED TO A TREATISE

ON

THE MOTION OF A RIGID BODY.

BY

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EXAMPLES

OF THE

MOTION OF A RIGID BODY.

The references in the following solutions are to articles in the treatise which they are intended to follow.

SECTION I.

GEOMETRICAL PROPERTIES OF A RIGID BODY.

1. (a) If x', y' be the co-ordinates of m referred to axes originating in the centre and parallel to those to which x, y refer,

$$x = a + x', \quad y = b + y'.$$

$$\therefore \Sigma(mxy) = \Sigma(m) \,.\, ab + a\Sigma(my') + b\Sigma(mx') + \Sigma(mx'y').$$

Now

$$\left.\begin{array}{l}\Sigma(mx') = 0\\ \Sigma(my') = 0\end{array}\right\},$$

by properties of the centre of gravity.

Also $\Sigma(mx'y') = 0$, since for any given value of y' the values of x' form pairs equal in magnitude and opposite in sign.

$$\therefore \Sigma(mxy) = \Sigma(m) \,.\, ab = Mab.$$

Since $z = 0$ for every point of the mass,

$$\therefore \Sigma(myz) = 0, \quad \Sigma(mxz) = 0.$$

The same method applies to (β), (ϵ), (ζ).

(γ) Since $z = 0$ for every point of the mass,

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0.$$

Let $OB = a$, $OA = b$. (fig. 1).

If P be a point of the lamina whose co-ordinates are x, y, the area of an element at P, whose sides are parallel to the axes of co-ordinates, is $dx.dy$, and this may also represent the mass of a corresponding element of the lamina if the mass of an unit of area of the lamina be unity;

$$\therefore \ \Sigma(mxy) = \int_x \int_y xy.$$

Now first the summation of the values of this function for elements which lie along the line MQ and form an elementary strip of the body in that direction, is equivalent to integrating with respect to y from $y = 0$ to $y = MQ = (a - x)\tan B$.

Hence for such a strip $\Sigma(myx)$ becomes

$$\tfrac{1}{2}\int_x x(a - x)^2 \tan^2 B$$

$$= \tfrac{1}{2}\int_x (a^2 x - 2ax^2 + x^3)\tan^2 B.$$

Secondly, the required summation will be completed by adding the values of the function for such strips as those just considered, as they range from OA to B, and this amounts to integrating the expression just obtained from $x = 0$ to $x = a$;

$$\therefore \text{ finally, } \Sigma(mxy) = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right) a^4 \tan^2 B$$

$$= \tfrac{1}{24} a^2 b^2;$$

while M, the mass, has by virtue of the units adopted been represented by $\frac{1}{2}ab$;

$$\therefore \ \Sigma(mxy) = \tfrac{1}{12} Mab.$$

(δ) Let l, m, n be the cosines of the inclinations of this line to the co-ordinate axes; r the distance of a point x, y, z in the line from its centre, and let the mass of an element at this point be represented by its length dr.

The co-ordinates of the centre of the line being

$$\frac{a + a'}{2}, \quad \frac{b + b'}{2}, \quad \frac{c + c'}{2},$$

we have

$$\Sigma(myz) = \Sigma\left(\frac{b+b'}{2} + rm\right)\left(\frac{c+c'}{2} + rn\right) dr$$

$$= M \cdot \frac{b+b'}{2} \cdot \frac{c+c'}{2} + mn \int_r^{-R,\,R} r^2,$$

if $2R$ be the whole length of the line,

$$= M \cdot \frac{b+b'}{2} \cdot \frac{c+c'}{2} + \tfrac{2}{3} mn \cdot R^3,$$

$$= M\left\{\frac{(b+b')(c+c')}{4} + \frac{(b-b')(c-c')}{12}\right\},$$

$$= \tfrac{1}{12} M \{4(bc + b'c') + 2(bc' + b'c)\},$$

$$= \tfrac{1}{6} M \{2(bc + b'c') + bc' + b'c\}.$$

So $\Sigma(mxz) = \tfrac{1}{6} M \{2(ac + a'c') + ac' + a'c\}.$

$$\Sigma(mxy) = \tfrac{1}{6} M \{2(ab + a'b') + ab' + a'b\}.$$

(η) This is an instance where the evaluation of a function is expedited by transforming it to other co-ordinate axes.

Let x', y' be the co-ordinates of the point x, y, when referred to other rectangular axes in the plane of xy so that x' is in the axis of the cone;

$$\therefore \left.\begin{aligned} x &= x' \cos a - y' \sin a \\ y &= x' \sin a + y' \cos a \end{aligned}\right\}.$$

The facility of the limits of integration in x' and y' recommends them in preference to x and y. Changing the integration to the former, we have

$$\Sigma(mxy) = \int_x \int_y xy$$

$$= \int_{x'} \int_{y'} xy \left(\frac{dx}{dx'} \cdot \frac{dy}{dy'} - \frac{dx}{dy'} \cdot \frac{dy}{dx'}\right)$$

(Gregory's *Ex.* Chap. III., or Moigno, Vol. II. p. 214)

$$= \int_{x'} \int_{y'} xy$$

$$= \int_{x'} \int_{y'} (x'^2 - y'^2) \,.\, \sin a \cos a$$

$$= \int_{x'} \int_{y'} \left\{ (x'^2 + z'^2) - (y'^2 + z'^2) \right\} \sin a \,.\, \cos a.$$

Now $\int_{x'} \int_{y} (x'^2 + z'^2)$ is the moment of inertia of the body about a line through its vertex perpendicular to its axis, and therefore $= \frac{3}{20} M\, (4a^2 + r^2)$, (Walton's *Examples.*)

and $\int_{x} \int_{y'} (y'^2 + z'^2)$ is the moment of inertia of the cone about its axis and $= \frac{3}{10} M r^2$,

$$\therefore \; \Sigma(mxy) = M \sin a \cos a \left\{ \tfrac{3}{5}\, a^2 - \tfrac{3}{20}\, r^2 \right\}.$$

2. If affixes denote quantities belonging to different particles of the body,

$\Sigma(my^2)\Sigma(mz^2)$ when developed consists of

(1) terms expressed by $\Sigma(m_1{}^2 y_1{}^2 z_1{}^2)$

produced by the multiplication of terms in the two factors belonging to the same particle,

(2) terms expressed by $\Sigma m_1 m_2 (y_1{}^2 z_2{}^2 + z_1{}^2 y_2{}^2)$,

produced by the union of terms belonging to different particles.

Again, $(\Sigma myz)^2$ will similarly be represented by

$$\Sigma(m_1{}^2 y_1{}^2 z_1{}^2) + 2\Sigma(m_1 m_2 y_1 z_1 y_2 z_2),$$

therefore the difference is $\Sigma m_1 m_2 (y_1 z_2 - z_1 y_2)^2$, a quantity necessarily positive.

$$\therefore \; \Sigma(my^2) \,.\, \Sigma(mz^2) > (\Sigma myz)^2.$$

So
$$\Sigma(mx^2)\Sigma(mz^2) > (\Sigma mxz)^2,$$
$$\Sigma(mx^2)\Sigma(my^2) > (\Sigma mxy)^2.$$

3. Let the arc AB (fig. 2) be doubled; then the moment of inertia of the whole BAC, whose mass is $2M$, will be double that of AB, which is required.

Let G be the centre of gravity of BAC.

Then $2Mr^2$ being the moment of inertia of BAC about an axis perpendicular to its plane through O the centre,

$$\text{moment required} = \tfrac{1}{2}(2Mr^2 - 2M.OG^2 + 2M.AG^2), \quad (8).$$
$$= Mr^2 - Mr.(OG - AG),$$
$$= 2Mr^2 - 2Mr.OG,$$
$$= 2Mr^2 - 2Mr^2\frac{\sin\theta}{\theta}.$$

The mode of obtaining the result in this question exemplifies the manner of finding the moment of inertia when that about another parallel axis is known, viz. by passing through a parallel axis at the centre of gravity. The moment of inertia of the axis through O being known by inspection, we get in succession: (1) that about a parallel axis at G, (2) that about a parallel axis at A.

4. The moment of inertia of the arc of the circle about an axis through its centre perpendicular to its plane is Mr^2, every particle having the same distance r from this axis. But this moment of inertia is the sum of those about two perpendicular diameters of the circle (9), and these latter are equal to one another, therefore either of them $= \frac{1}{2}Mr^2$.

5. Let α, β be the semi-axes of one stratum. A homogeneous spheroid bounded by this surface will have about its axis of figure the moment of inertia $\frac{8}{15}\pi\rho\alpha^4\beta$, ρ being its density, or $\frac{8}{15}\pi\rho\alpha^5\sqrt{1-e^2}$.

Now if this expression be differentiated with regard to a, ρ and e remaining unaffected, we obtain the moment of inertia of an infinitely thin homogeneous shell between the surface above mentioned and a contiguous one. The result is

$$\tfrac{8}{3}\pi\rho a^4\sqrt{1-e^2}\,.\,da.$$

The integral of this from zero to a is the moment of inertia of the body formed by an assemblage of such shells, ρ being a function of a in this integration according to the law of density of the solid.

Ex. $${}^{0}\!\int_{\alpha}^{a} \rho a^4 = c\,{}^{0}\!\int_{\alpha}^{a} a^3 \sin n a$$

$$= -\frac{c}{n}\,{}^{0}\!\int_{\alpha}^{a} a^3 \frac{d(\cos n a)}{d a}$$

$$= -\frac{c}{n} a^3 \cos n a + \frac{3c}{n}\,{}^{0}\!\int_{\alpha}^{a} a^2 \cos n a.$$

$${}^{0}\!\int_{\alpha}^{a} a^2 \cos n a = \frac{1}{n}\,{}^{0}\!\int_{\alpha}^{a} a^2 \frac{d(\sin n a)}{d a}$$

$$= \frac{1}{n} a^2 \sin n a - \frac{2}{n}\,{}^{0}\!\int_{\alpha}^{a} a \sin n a\,;$$

$${}^{0}\!\int_{\alpha}^{a} a \sin n a = -\frac{1}{n} a \cos n a + \frac{1}{n^2} \sin n a\,;$$

$$\therefore\ {}^{0}\!\int_{\alpha}^{a} a^2 \cos n a = \frac{a^2}{n} \sin n a + \frac{2a}{n^2} \cos n a - \frac{2}{n^3} \sin n a,$$

$${}^{0}\!\int_{\alpha}^{a} \rho a^4 = -\frac{c a^3}{n} \cos n a + \frac{3 c a^2}{n^2} \sin n a + \frac{6 c a}{n^3} \cos n a - \frac{6c}{n^4} \sin n a.$$

Therefore the moment of inertia required is

$$\frac{8}{3}\,\frac{\pi c\sqrt{1-e^2}}{n^4}\left\{3(a^2 n^2 - 2)\sin n a + (6 a n - a^3 n^3)\cos n a\right\}.$$

6. If following the notation of (10) we denote quantities relative to the different axes of the system by affixes, since

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$
$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1,$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1;$$
$$\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0,$$
$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0,$$
$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$
$$\therefore\ Q_1 + Q_2 + Q_3 = A + B + C,$$

a result independent of the particular axes in question.

7. Let x', y', z' be co-ordinates of a particle m referred to co-ordinate axes through a point in the axis of z at distance γ from the origin of the co-ordinates x, y, z, the axes of z and z' being the same, and the axes of x', y' making angles θ with the axes of x and y respectively.

$$\begin{aligned}\Sigma(my'z') &= \Sigma m(x\sin\theta + y\cos\theta)(z-\gamma)\\ &= -\gamma\sin\theta\,\Sigma(mx) - \gamma\cos\theta\,\Sigma(my),\end{aligned}$$

since $\Sigma(mxz) = 0$, $\Sigma(myz) = 0$ by virtue of the properties of principal axes.

$$\Sigma(mx'z') = -\gamma\cos\theta\,\Sigma(mx) + \gamma\sin\theta\,\Sigma(my),$$
$$\Sigma(mxy) = \Sigma m(x^2 - y^2)\sin\theta\cos\theta + \Sigma(mxy)(\cos^2\theta - \sin^2\theta).$$

Now the last of these expressions can in general be reduced to zero by assuming θ properly; but after θ is thus assigned the two former expressions will not in general also vanish for a finite value of γ unless $\Sigma(mx) = 0$, $\Sigma(my) = 0$, or unless the axis of z passes through the centre of gravity of the body.

8. Let x', y', z' be the co-ordinates of m referred to a system of rectangular axes whose direction cosines in reference to the given axes are

$$l,\ m,\ n;\quad l',\ m',\ n';\quad l'',\ m'',\ n''.$$

$$\begin{aligned}\Sigma(my'z') &= \Sigma m\,(l'x + m'y + n'z)\,(l''x + m''y + n''z),\\ &= l'l''\,\Sigma(mx^2) + m'm''\,\Sigma(my^2) + n'n''\,\Sigma(mz^2),\\ &= (l'l'' + m'm'' + n'n'')\,\Sigma(mx^2),\\ &= 0.\end{aligned}$$

So $\Sigma(mx z') = 0$;

$\Sigma(mx'y') = 0$;

or the new system is one of principal axes.

9. With the notation of (17)

$$\tan 2\theta = \frac{2\,\Sigma(mxy)}{\Sigma(mx^2) - \Sigma(my^2)}.$$

Now $\Sigma(mxy) = \frac{1}{12}Mab$, (supra, p. 2)

$$\begin{aligned}\Sigma(mx^2) &= \tfrac{1}{3}\int_y \frac{a^3}{b^3}(b-y)^3,\\ &= \tfrac{1}{12}a^3b,\\ &= \tfrac{1}{6}Ma^2.\end{aligned}$$

$$\Sigma(my^2) = \tfrac{1}{6}Mb^2;$$

$$\therefore\ \tan 2\theta = \frac{ab}{a^2 - b^2},$$

$$\theta = \tfrac{1}{2}\tan^{-1}\frac{ab}{a^2 - b^2}.$$

10. If x', y' be the co-ordinates of a particle m referred to the centre, so that

$$x = x' - \alpha,\quad y = y' - \beta,$$

$$\begin{aligned}\Sigma(mxy) &= M\alpha\beta,\\ \Sigma(mx^2) &= \Sigma(mx'^2) + M\alpha^2,\\ &= \tfrac{1}{4}Ma^2 + M\alpha^2,\\ \Sigma(my^2) &= \tfrac{1}{4}Mb^2 + M\beta^2,\end{aligned}$$

$$\therefore\ \tan 2\theta = \frac{2\alpha\beta}{\frac{1}{4}(a^2 - b^2) + \alpha^2 - \beta^2}.$$

11. Since every system through the centre of the cube is one of principal axes, the moment of inertia about every axis though the centre of the cube is the same.

$\therefore$ moment of inertia about the diagonal

= moment of inertia about a line through the centre parallel to four of the edges,

$= \frac{1}{6} M a^2.$

Again, moment about the diagonal of a face,

$= \frac{1}{4} M a^2 +$ moment about a parallel line through the centre,

$= \frac{1}{4} M a^2 + \frac{1}{6} M a^2,$

$= \frac{5}{12} M a^2.$

12. This example and those which follow it shew how the moment of inertia about any line through a point can be obtained, if the principal moments at that point are known.

Let α be the semi-vertical angle of the cone. Then its moments of inertia about its axis and about a line through its vertex perpendicular to its axis are

$$\left.\begin{array}{r}\frac{3}{10} M c^2 \\ \text{and } \dfrac{3M}{20}(4a^2 + c^2)\end{array}\right\}. \quad \text{(Walton's } \textit{Examples.)}$$

Therefore moment of inertia required

$$= M\left\{\tfrac{3}{10} c^2 . \cos^2 \alpha + \tfrac{3}{20}(4a^2 + c^2)\sin^2 \alpha\right\} \qquad (21),$$

$$= \frac{3M}{20}\left\{\frac{2a^2c^2 + (4a^2 + c^2)c^2}{a^2 + c^2}\right\},$$

$$= \tfrac{3}{20} M c^2 . \frac{6a^2 + c^2}{a^2 + c^2}.$$

13. If a, b be the semi-axes of an ellipse, its moments of inertia about its axes are $\frac{1}{4} M b^2$ and $\frac{1}{4} M a^2$.

Since these are principal axes, the moment of inertia about CP any diameter inclined at an angle α to the major axis is

$$\tfrac{1}{4} M \left\{ b^2 \cos^2 \alpha + a^2 \sin^2 \alpha \right\}. \quad (21)$$

$$\text{Now } \frac{1}{CP^2} = \frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2};$$

$$\therefore\ b^2 \cos^2 \alpha + a^2 \sin^2 \alpha \propto \frac{1}{CP^2}.$$

14. Since the axes of an ellipsoid are principal axes, and the moments of inertia about them are

$$M \frac{b^2 + c^2}{5}, \quad M \frac{a^2 + c^2}{5}, \quad M \frac{a^2 + b^2}{5},$$

hence the moment of inertia about a diameter whose direction cosines relatively to these are l, m, n, is

$$\frac{M}{5} \left\{ (b^2 + c^2)\, l^2 + (a^2 + c^2)\, m^2 + (a^2 + b^2)\, n^2 \right\},$$

$$= \frac{M}{5} \left\{ a^2 + b^2 + c^2 - a^2 l^2 - b^2 m^2 - c^2 n^2 \right\}.$$

Now if the tangent plane perpendicular to this diameter whose equation is

$$lx + my + nz = p,$$

touches the ellipsoid at x', y', z', this equation must be identical with

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1;$$

$$\therefore\ \frac{x'}{a^2} = \frac{l}{p}, \quad \frac{y'}{b^2} = \frac{m}{p}, \quad \frac{z'}{c^2} = \frac{n}{p}.$$

$$\therefore\ 1 = \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 + \left(\frac{z'}{c}\right)^2 = \frac{a^2 l^2 + b^2 m^2 + c^2 n^2}{p^2};$$

$$\therefore\ \text{moment required} = \frac{M}{5} \left\{ a^2 + b^2 + c^2 - p^2 \right\}.$$

15, 16. If A be the moment of inertia of a solid of revolution about its axis of figure, B that about a principal axis perpendicular to this axis, then the moment of the body about a line through the intersection of these two inclined at the $\angle\, \alpha$ to the axis of figure is

$$A \cos^2 \alpha + B \sin^2 \alpha,$$

$$\text{or} \quad A + (B - A) \sin^2 \alpha.$$

Therefore if the moment of inertia be the same for all such axes, or be independent of α, $B = A$.

Hence we must have in the former example

$$\tfrac{2}{5} a^2 = \tfrac{1}{5} (a^2 + b^2) + b^2,$$

$$a^2 = 6b^2,$$

$$a = b\sqrt{6},$$

a and b being the half axes of the solid.

In the latter example if a be the altitude, c the radius of the cone's base,

$$\frac{3}{10} c^2 = \frac{3}{20}(4a^2 + c^2),$$

$$c^2 = 4a^2,$$

$$a = \frac{c}{2}.$$

SECTION II.

MOTION OF A BODY ABOUT A FIXED AXIS.

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1. The angular velocity

$$\frac{d\theta}{dt} = a \sec^2 \theta ;$$

$$\therefore \frac{dt}{d\theta} = \frac{1}{a} \cos^2 \theta,$$

$$= \frac{1}{2a} (1 + \cos 2\theta).$$

$$\therefore t = C + \frac{1}{2a} \left\{ \theta + \frac{\sin 2\theta}{2} \right\};$$

therefore if the integral be taken between values of θ separated by the interval 2π, we have

$$\text{time of revolution} = \frac{2\pi}{2a} = \frac{\pi}{a};$$

whereas the time of revolution if the angular velocity a were uniform would be $\dfrac{2\pi}{a}$.

2. At time t the distance of the point x, y, z from the axis of rotation is

$$\rho = \frac{\{x^2 + y^2 + z^2 - (lx + my + nz)^2\}^{\frac{1}{2}}}{\sqrt{l^2 + m^2 + n^2}}.$$

Let the differentials dx, dy, dz denote the spaces through which the point moves in directions of the co-ordinate axes in the elementary time dt. Now by the nature of rotatory motion the distance of the point x, y, z from the axis remains

constant, as well as its distance from the origin, which conditions require that

$$x^2 + y^2 + z^2$$

$$\text{and } lx + my + nz$$

should remain invariable;

$$\therefore\ x\,dx + y\,dy + z\,dz = 0,$$

$$l\,dx + m\,dy + n\,dz = 0.$$

By elimination between these we have

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = \lambda \text{ suppose};$$

$$\therefore\ (dx)^2 + (dy)^2 + (dz)^2 = \lambda^2 \left\{x^2 + y^2 + z^2 - (lx + my + nz)^2\right\}.$$

Now $\rho\omega \,.\, dt$ will ultimately represent the space through which the point moves in the time dt

$$= \sqrt{(dx)^2 + (dy)^2 + (dz)^2},$$

$$= \lambda \sqrt{x^2 + y^2 + z^2 - (lx + my + nz)^2},$$

$$= \lambda\rho \sqrt{l^2 + m^2 + n^2};$$

$$\therefore\ \omega\,dt = \lambda \sqrt{l^2 + m^2 + n^2};$$

$$\therefore\ \frac{dx}{dt} = \frac{\lambda}{dt}(mz - ny),$$

$$= \frac{mz - ny}{\sqrt{l^2 + m^2 + n^2}}\,\omega;$$

$$\frac{dy}{dt} = \frac{nx - lz}{\sqrt{l^2 + m^2 + n^2}}\,\omega;$$

$$\frac{dz}{dt} = \frac{ly - mx}{\sqrt{l^2 + m^2 + n^2}}\,\omega.$$

These are the velocities of the point parallel to the co-ordinate axes.

3. The moment of inertia of a solid hemisphere of radius r about a diameter of its base is

$$\frac{2\pi r^3}{3} \cdot \tfrac{2}{5} r^2 = \tfrac{4}{15} \pi r^5, \text{ the density being } 1;$$

therefore if dr be the thickness of an indefinitely thin shell, its moment of inertia is $\frac{4}{3} \pi r^4 . dr$.

Again, the moment of the solid hemisphere about its base is

$$\frac{2\pi r^3}{3} \cdot \tfrac{3}{8} r = \tfrac{1}{4} \pi r^4;$$

$$\text{therefore that of the shell is } \pi r^3 . dr;$$

$$\text{therefore length of simple pendulum} = \frac{\frac{4}{3}\pi r^4 dr}{\pi r^3 dr}, \quad (49)$$

$$= \frac{4r}{3}.$$

$$\text{and time of small oscillation} = \pi \sqrt{\frac{4r}{3g}},$$

the unit of time being that assumed in measuring numerically the force of gravity g.

4. If O be the centre of the arc, (fig. 2) A its middle point, G its centre of gravity, and M its mass, $OA = a$, its moment of inertia about an axis through A perpendicular to its plane

$$= M(k^2 + AG^2), \quad (6)$$

$$= M(k^2 + OG^2 + AG^2 - OG^2),$$

$$= Ma^2 + Ma(AG - OG),$$

$$= Ma^2 + Ma(2AG - a)$$

$$= 2Ma . AG;$$

therefore the length of the simple equivalent pendulum

$$= 2a; \quad (49)$$

$$\therefore \pi \sqrt{\frac{2a}{32 \cdot 2}} = 1 \cdot 5708,$$

$$a = \tfrac{1}{2} \left(\frac{1 \cdot 5708}{3 \cdot 1416}\right)^2 . 32 \cdot 2,$$

$$= 4 \text{ nearly.}$$

Since 32·2 has been adopted as the measure of the force of gravity with a second for the unit of time, the unit of length is a foot.

5. Let C be the middle point of AB the vertical chord, O the centre of the sphere. (fig. 3).

If ω be the angular velocity of the body, the effective force on a particle of mass m at distance r from the axis is $m\omega^2 r$ towards the axis, and the resultant of such is $M\omega^2 . CO$ in OC. Hence if this force be applied in the reversed direction CO, there will be equilibrium between it, the weight, and the reaction of the axis. Since the two former forces are in the plane ABO, the latter reaction is in this plane also, and may be represented by

$$\left.\begin{matrix} F \\ G \end{matrix}\right\} \text{ at } A, \text{ and } \left.\begin{matrix} F' \\ G' \end{matrix}\right\} \text{ at } B, \quad \begin{matrix} \text{horizontal} \\ \text{vertical.} \end{matrix}$$

$$\therefore O = F + F' + M\omega^2 . CO,$$

$$O = G + G' - Mg, \qquad (30)$$

$$O = F . AC + Mg . CO - F' . BC;$$

the former equations arising from horizontal and vertical resolution, the last from taking moments about an axis through C perpendicular to plane OAB;

$$\therefore O = 2F . AC + M\omega^2 . AC . CO + Mg . CO,$$

$$O = 2F' . AC + M\omega^2 . AC . CO - Mg . CO.$$

These equations assign F and F, while the sum of G and G' is already known, the separate values of G and G' being indeterminate.

Now if the chord becomes a tangent, $AC = 0$, and F, F' become infinite. The interpretation of this is that motion of the kind considered cannot exist while the sphere is attached to the axis at *one* point only.

6. Let C (fig. 4) be the middle point of AB, the vertical side about which the square revolves, G its centre of gravity. Let the reasoning of the last question be followed, and the reaction of the axis thus reduced to the

$$\begin{matrix}\text{horizontal}\\ \text{vertical}\end{matrix}\ \text{forces}\ \left.\begin{matrix}F\\ G\end{matrix}\right\}\ \text{at } A, \text{ and } \left.\begin{matrix}F'\\ G'\end{matrix}\right\}\ \text{at } C.$$

$$\therefore\ O = F + F' + M\omega^2 . \frac{AB}{2},$$

$$O = G + G' - Mg,$$

$$O = F . AC + Mg . \frac{AB}{2}.$$

(1) If the pressure on the axis acts entirely at A,

$$F' = 0,\ G' = 0.$$

$$\therefore\ F = -Mg,$$

$$F = -M\omega^2 . \frac{AB}{2}.$$

$$\therefore\ \omega^2 . AB = 2g;$$

under which condition the supposition is possible.

(2) If the pressure act at C, $F = 0$, $G = 0$,

$$\therefore\ Mg = 0,$$

which is impossible.

7. The pressure upon an axis can be determined by obtaining its resolved parts (1) in directions fixed in space (42), (2) in directions fixed relatively to the body but moveable in space (46). The latter method is generally preferable,

by reason of the difficulty which often arises in evaluating the function contained in the results of (42), and the following two examples are instances of it. According to the notation of (46) if ϕ measures the inclination of the rod to the horizon,

$$(k^2 + \bar{r}^2)\frac{d^2\phi}{dt^2} = g\bar{r} \,.\, \cos\phi,$$

$$(k^2 + \bar{r}^2)\left(\frac{d\phi}{dt}\right)^2 = 2g\bar{r} \,.\, (\sin\phi - \sin\alpha),$$

if α be the inclination of the rod to the horizon when it begins to move.

Now (1) if the pressure on the axis perpendicular to the rod is zero,

$$0 = Q - M\bar{r}\frac{d^2\phi}{dt^2}, \tag{46}$$

$$0 = g\cos\phi - \frac{g\bar{r}^2}{k^2 + \bar{r}^2}\cos\phi\,;$$

$$\therefore\ \cos\phi = 0, \text{ or the rod is vertical.}$$

(2) if the pressure along the rod vanishes,

$$0 = g\sin\phi + \frac{2g\bar{r}^2}{k^2 + \bar{r}^2}(\sin\phi - \sin\alpha),$$

$$0 = \sin\phi + \frac{2}{1 + \frac{1}{3}}(\sin\phi - \sin\alpha),$$

$$= \sin\phi + \tfrac{3}{2}(\sin\phi - \sin\alpha),$$

$$0 = 5\sin\phi - 3\sin\alpha,$$

$$5\sin\phi = 3\sin\alpha.$$

8. Let a = length of axis,

r = radius of base;

$$\therefore\ k^2 + \frac{a^2}{4} = \frac{a^2}{3} + \frac{r^2}{4}\,;$$

therefore with the notation of (46),

$$\left(\frac{a^2}{3}+\frac{r^2}{4}\right)\frac{d^2\phi}{dt^2}=g\cos\phi\,.\,\frac{a}{2},$$

$$\left(\frac{a^2}{3}+\frac{r^2}{4}\right)\left(\frac{d\phi}{dt}\right)^2=2g\sin\phi\,.\,\frac{a}{2};$$

$$\text{or }\frac{d^2\phi}{dt^2}=\frac{\frac{a}{2}}{\frac{a^2}{3}+\frac{a^2}{6}}g\cos\phi=\frac{g}{a}\cos\phi;$$

$$\left(\frac{d\phi}{dt}\right)^2=\frac{2g}{a}\sin\phi;$$

$$\therefore\ R=Mg\sin\phi+Mg\sin\phi=2\,Mg\sin\phi,$$

$$S=Mg\cos\phi-\tfrac{1}{2}Mg\cos\phi=\tfrac{1}{2}Mg\cos\phi;$$

$$\therefore\ \tan(\psi-\phi)=\frac{S}{R}=\tfrac{1}{4}\cot\phi;$$

$$4\tan(\psi-\phi)=\cot\phi.$$

9. In the position contemplated the effective force on a molecule m whose distance from the axis is r is $m\omega^2r$ towards the axis, and this can be replaced by

$$m\omega^2x \text{ horizontally},$$

$$m\omega^2y \text{ vertically}.$$

Let the horizontal reaction of the axis be represented by a force F at the origin, and another F' at a distance a.

$$\therefore\ \left.\begin{aligned}0&=F+F'+\Sigma(m\omega^2x)\\0&=F'a+\Sigma(m\omega^2xz)\end{aligned}\right\}.$$

Now $\Sigma(mx)=0$; $\therefore$ $F+F'=0$, but the forces F, F' will not separately vanish unless $\Sigma(mxz)=0$. If the form of the body does not satisfy this condition, the horizontal pressure on the axis forms a couple.

9*. The rotatory motion of the hemisphere will be caused by the part $g \sin \alpha$ of the force of gravity, which acts perpendicularly to the axis. Hence if ϕ be the angle which the base makes with its original position, $\bar{r}$ the distance of the centre of gravity of the body from the axis,

$$(k^2 + \bar{r}^2) \frac{d^2\phi}{dt^2} = g\bar{r} \sin \alpha \,.\, \cos \phi,$$

$$(k^2 + \bar{r}^2) \left(\frac{d\phi}{dt}\right)^2 = 2g\bar{r} \sin \alpha \,.\, \sin \phi;$$

therefore if ω be the angular velocity in the lowest position when $\phi = \frac{\pi}{2}$,

$$\omega^2 = \frac{2g\bar{r} \sin \alpha}{k^2 + \bar{r}^2}.$$

In the same position, the resultant effective force is $M\omega^2\bar{r}$ towards the axis,

$$= \frac{2\bar{r}^2}{k^2 + \bar{r}^2} Mg \sin \alpha,$$

$$= \frac{2 \,.\, (\frac{3}{8})^2}{\frac{2}{5}} Mg \sin \alpha,$$

$$= \tfrac{45}{64} Mg \sin \alpha.$$

Hence the pressure on the axis in direction of its length

$$= Mg \cos \alpha;$$

and that perpendicular to its length

$$= Mg \sin \alpha + \tfrac{45}{64} Mg \sin \alpha,$$

$$= \tfrac{109}{64} Mg \sin \alpha;$$

therefore the resultant of these two

$$= \tfrac{1}{64} \sqrt{(109 \sin \alpha)^2 + (64 \cos \alpha)^2} \times \text{weight of the body.}$$

10. If we investigate the position of the axis which gives the minimum time of oscillation among all axes in its

own direction, we have to make $\frac{k^2 + h^2}{h}$ a minimum by the variation of h (49).

$$\therefore\ h^2 = k^2,$$

and the time of such minimum oscillation $= \pi \sqrt{\frac{2k}{g}}$.

Hence among all such modes of oscillation, each the minimum in reference to its own direction, we have furthermore to select the minimum consequent on the variation of k. To obtain this the direction must be that of the principal axis of least moment, i. e. a diameter of the spheroid's equator. If a, c be the semi-axes of the spheroid, the axis required is at a distance $\sqrt{\frac{2}{g}} \cdot \sqrt{\frac{a^2 + c^2}{5}}$ from a parallel diameter of the equator.

11. Let a be the length of an edge of the cube.

If ω be its angular velocity in its lowest position, the pressure on the axis in that position

$$= Mg + M\omega^2 \frac{a}{\sqrt{2}}.$$

This pressure will therefore be least when ω has the least value compatible with the condition of the body making complete revolutions, or having a finite velocity in the highest position.

The equation of motion of the body is

$$\tfrac{2}{3} a^2 \frac{d^2\phi}{dt^2} = \frac{a}{\sqrt{2}} g \cos\phi,$$

$$\therefore\ \tfrac{2}{3} a^2 \left(\frac{d\phi}{dt}\right)^2 = C + \sqrt{2} \,.\, ag \cos\phi.$$

The limit of ω will result from making $\frac{d\phi}{dt} = 0$, when $\phi = -\frac{\pi}{2}$.

$$\therefore \; \tfrac{2}{3} a\omega^2 = 2\sqrt{2} \,.\, g,$$

and in this case the pressure on the axis

$$= Mg + 3Mg = 4 \,.\, \text{weight of the body},$$

therefore the axis must be able to sustain this pressure at least.

12. The pressure on the axis in the lowest position of the body is $Mg + Mr\omega^2$. But if the whole were collected at the centre of oscillation, since the distance of this point from the axis $= (\frac{2}{5} + 1)r$, the pressure would be $Mg + Mr\omega^2 + \frac{2}{5}Mr\omega^2$,

$$\therefore \text{ difference} = \tfrac{2}{5} Mr\omega^2.$$

This example is meant to guard the reader against supposing that because the angular motion is the same in the given body as if all its mass were collected at the centre of oscillation, other circumstances also, such as the pressure on the axis, may be considered the same in the two cases.

13. Let AB, AC (fig. 5) be the rods equally inclined to the vertical AD.

The effective force on any molecule m of either rod at distance r from the axis is $mr\omega^2$, acting towards AD, and if such forces be reversed, the moment of their resultant about A when the rod AB is considered

$= \Sigma(mr^2 \cot\alpha)\omega^2$,

$= \cot\alpha \sin^2\alpha \,.\, \omega^2 \times$ moment of inertia of AB about a perpendicular axis at A,

$$= M \,.\, \frac{AB^2}{3} \,.\, \omega^2 \cos\alpha \sin\alpha.$$

Therefore if moments be taken about A, and if T be the tension of BC,

$$T \,.\, AD = M \,.\, \frac{AB^2}{3}\, \omega^2 \cos\alpha \sin\alpha + Mg \,.\, \frac{AB}{2} \sin\alpha,$$

$$T \,.\, \cos\alpha = \tfrac{1}{3} M \,.\, AB \,.\, \omega^2 \,.\, \cos\alpha \sin\alpha + \tfrac{1}{2} Mg \,.\, \sin\alpha,$$

$$T = \tfrac{1}{2} Mg \tan\alpha + \tfrac{1}{3} Ma\omega^2 \,.\, \sin\alpha.$$

14. Let M, m be the masses of the cylinder and the rod, and at a certain given temperature let r be the radius of the cylinder, a its axis, b the length of the rod.

If L be the length of the equivalent simple pendulum,

$$L\left\{M\left(b-\frac{a}{2}\right)+m\,.\,\frac{b}{2}\right\}=M\,.\left\{\frac{r^2}{4}+\frac{a^2}{12}+\left(b-\frac{a}{2}\right)^2\right\}+m\,.\,\frac{b^2}{3}. \quad (1)$$

Now by a change of temperature let

$$\left.\begin{matrix} a \\ r \end{matrix}\right\} \text{ become } \left\{\begin{matrix} a(1+e), \\ r(r+e), \end{matrix}\right.$$

$$b \;\ldots\ldots\ldots\; b(1+e'),$$

e and e' being the expansibilities of the substances for the given change of temperature. Then if L be unchanged, neglecting the squares of e and e' we have

$$L\left\{M.(e'b-\tfrac{1}{2}ea)+m.\tfrac{1}{2}e'b\right\}$$

$$=2M\left\{\frac{er^2}{4}+\frac{ea^2}{12}+\left(b-\frac{a}{2}\right)\left(e'b-\frac{ea}{2}\right)\right\}+m\frac{e'b^2}{3}. \quad (2)$$

If L be eliminated by division between (1) and (2), the resulting relation connects the masses, the dimensions, and expansibilities of the two parts of the compound body.

15. Let A (fig. 6) be the given point of the rod, whose ends are attached to the spheres at the points B, C. Let M, M' be the masses of the spheres B, C, and m, m' those of the portions of rod AB, AC. Also at some assigned temperature let r, r' be the radii of the spheres, a, a' lengths of AB, AC.

If L be the length of the simple equivalent pendulum,

$$L\left(Ma+m\frac{a}{2}-M'a'-m'\frac{a'}{2}\right)$$

$$=(\tfrac{2}{5}r^2+a^2)M+\tfrac{1}{3}ma^2+(\tfrac{2}{5}r'^2+a'^2)M'+\tfrac{1}{3}m'a'^2. \quad (1)$$

Let a, a' become $a(1+\epsilon)$, $a'(1+\epsilon)$ and r, r' become $r(1+e)$, $r'(1+e')$ for a given change of temperature, the squares of the expansions being so small as to be neglected.

Therefore if L remain unaltered

$$\frac{L}{2}\left\{Ma + m\frac{a}{2} - M'a' - m'\frac{a'}{2}\right\}\epsilon$$

$$= (\tfrac{2}{5}r^2e + a^2\epsilon)M + \tfrac{1}{3}ma^2\epsilon + (\tfrac{2}{5}r'^2e' + a'^2\epsilon)M' + \tfrac{1}{3}m'a'^2\epsilon. \quad (2)$$

If L be eliminated between (1) and (2), the resulting relation connects the masses, dimensions, and expansibilities of the parts.

16. If k, k' be the radii of gyration of the bodies about lines through their own centres of gravity parallel to the axis of suspension, (fig. 7),

$$L(mh + m'h') = m(h^2 + k^2) + m'(h'^2 + k'^2).$$

Now after the rise of temperature, k, h, k' become $k(1 + r)$, $h(1 + r)$, $k'(1 + r')$, but the augmentation of h' arises from two causes, (1) the change of SP, (2) the change of size of the lower body,

$$\therefore\ h' \text{ becomes } e(1 + r) + (h' - e)(1 + r'),$$

$$\text{or } h' + h'r' + e(r - r').$$

Therefore if L be permanent,

$$\frac{L}{2}\{mrh + m'r'h' + m'e(r - r')\}$$

$$= m(h^2 + k^2)r + m'k'^2r' + m'h'^2r' + m'h'e(r - r')$$

$$= mrhl + m'r'h'l' + m'h'e(r - r').$$

17. These examples (17) and (18) are placed together to induce the reader to observe why the action is impulsive in one of them and not in the other. In (18) finite motion is suddenly destroyed, and impulsive force alone can do this. In (17) the motion commences from rest, and increases gradually from that state, and finite force is adequate to produce this effect.

At time t let the rod make an angle ϕ with the horizon. Let $2a$ be its length.

$$\therefore\ (k^2 + a^2)\frac{d^2\phi}{dt^2} = ga\cos\phi\,;$$

$$\therefore \text{ when } \phi = 0,\quad \frac{d^2\phi}{dt^2} = \frac{ga}{a^2 + k^2}.$$

Now initially the effective force on a molecule m, at distance r from the fixed extremity is

$$mr\left(\frac{d^2\phi}{dt^2}\right)_{\phi=0} \quad \text{or} \quad mr\frac{ga}{a^2 + k^2},$$

acting downwards, and if this be applied upwards the resultant of such reversed forces, together with R the initial reaction at the fixed point, can balance the weight Mg of the rod.

$$\begin{aligned} \therefore\ Mg &= R + \Sigma(mr)\,.\frac{ga}{a^2 + k^2}, \\ &= R + \frac{a^2}{a^2 + k^2}Mg, \\ &= R + \frac{1}{1 + \frac{1}{3}}Mg, \\ &= R + \tfrac{3}{4}Mg. \end{aligned}$$

$$\therefore\ R = \tfrac{1}{4}Mg,$$

while before the other support was removed, the pressure was $\frac{1}{2}Mg$ on each point of attachment.

18. Let the rod AB, whose length is a (fig. 8), be swinging upwards about the point A, with the angular velocity ω, and be suddenly fixed at B and thus brought to rest in a horizontal position. Let r be the distance from A of a molecule m. Such a molecule is moving before the check with the velocity ωr, and therefore the effective impulse applied to it is $m\omega r$ downwards. If such effective impulses be reversed and supposed to act upwards on the rod, they can satisfy the con-

ditions of equilibrium with the reactions R, R' at A and B. Hence these reactions must be vertical.

If we now take the equation of moments about A and B in succession, we have

$$aR = \Sigma\{mr(a-r)\}\,\omega,$$

$$aR' = \Sigma(mr^2)\,\omega,$$

$$\therefore\ \frac{R}{R'} = \frac{a\Sigma(mr) - \Sigma(mr^2)}{\Sigma(mr)}$$

$$= \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{3}}$$

$$= \tfrac{1}{2}.$$

This solution and result will apply as well if the position of the rod when it is brought to rest is not horizontal.

19. Let the square be revolving about its side AB (fig. 9) with an angular velocity ω which is suddenly destroyed by a blow P at C. Let a be a side of the square.

Now if m be the mass of a molecule at distance r from the axis, this particle suddenly loses the velocity $r\omega$, and therefore the effective force upon it is $mr\omega$ parallel to P. These effective forces acting in a reversed direction, i.e. opposite to P, can maintain equilibrium with P and the reaction of the axis. Hence the latter reaction is perpendicular to the plane, and may be exhibited by two impulsive forces R, R' at A and B.

If moments be now taken about AB, AC, BD in succession, x being the distance of m from AC,

$$Pa = \Sigma(mr^2)\,\omega = M\,.\,\frac{a^2}{3}\,\omega, \tag{1}$$

$$R'a = \Sigma(mrx)\,.\,\omega = \tfrac{1}{4}Ma^2\omega, \tag{2}$$

$$(R+P)\,a = \Sigma\{mr(a-x)\}\,\omega = \tfrac{1}{4}Ma^2\omega. \tag{3}$$

Hence $P = \frac{1}{3} M a \omega$ the required impulse at C.

$$\left.\begin{aligned} R' &= \tfrac{1}{4} M a \omega \\ R &= -\tfrac{1}{12} M a \omega \end{aligned}\right\} \text{ the impulses on the axis.}$$

20. Let AB be the diameter perpendicular to the tangent line. The effective force has a resultant acting through some point in this line perpendicular to the plane, by reason of the symmetry of the figure and its motion about AB. If then the blow be applied at a point not in AB, this impulse will have an unbalanced moment about AB, and the conditions supposed are impossible.

21. Let AB (fig. 10) be the rod moving about A, and impinging at B on a vertical plane with an angular velocity ω, which is converted by the impulse R into the angular velocity ω' in the contrary direction. Let a be the rod's inclination to the horizon;

$$\therefore\ M\left(k^2 + \frac{AB^2}{4}\right)(\omega + \omega') = R \,.\, AB \,.\sin a.$$

Now if the rod had no elasticity, and R_1 were the impulse in this case, R_1 would destroy ω but generate no angular velocity again,

$$\therefore\ M\left(k^2 + \frac{AB^2}{4}\right)\omega = R_1 \,.\, AB \sin a;$$

$$\therefore\ \frac{\omega + \omega'}{\omega} = \frac{R}{R_1}.$$

$$\text{But}\quad R = R_1 + eR_1; \qquad (134)$$

$$\therefore\ \omega' = e \,.\, \omega.$$

22. For the solution of this problem we have to determine

(1) the angular velocity to be generated in the rod in order that it may rise into its position of unstable equilibrium;

(2) the impulse which may generate this angular velocity without causing any impulse on the axis.

Let ω be the angular velocity generated in the rod in its lowest position to make it rise into the position of unstable equilibrium. The equation of its motion after the blow will be

$$(a^2 + k^2)\frac{d^2\phi}{dt^2} = -ga\sin\phi,$$

ϕ being measured from the original distance of the rod.

$$\therefore\ (a^2 + k^2)\left(\frac{d\phi}{dt}\right)^2 = C + 2ga\cos\phi.$$

Now by the conditions of the question

$$(a^2 + k^2)\,\omega^2 = C + 2ga,$$

$$0 = C - 2ga;$$

$$\therefore\ \omega^2 = \frac{4ga}{a^2 + k^2} = \frac{4ga}{a^2 + \frac{1}{3}a^2} = \frac{3g}{a}.$$

Now let R be the blow which applied perpendicularly to the rod produces this angular velocity. The effective impulse applied to any molecule m at distance r from the axis is $mr\omega$, and such impulses reversed must balance R, because R is the only impressed force if the reaction of the axis is zero,

$$\therefore\ R = \Sigma(mr\omega) = Ma\omega,$$

$$= Ma\sqrt{\frac{3g}{a}},$$

$$= M\sqrt{3ag}.$$

23. Let a = radius of the circle,

x = distance from the axis of the point where the blow R acts.

Then by the method of the last example,

$$(a^2 + k^2)\,\omega^2 = 2ga,$$

$$R = Ma\omega,$$

$$Rx = M(a^2 + k^2)\,\omega;$$

$$\therefore \quad \omega^2 = \frac{2g}{(1+\frac{1}{4})\,a} = \frac{8}{5}\cdot\frac{g}{a}.$$

$$R = M\sqrt{\tfrac{8}{5}\,ag},$$

$$x = \frac{a^2+k^2}{a} = \tfrac{5}{4}\,a.$$

24. Let AB (fig. 11) be the diagonal about which the rectangle can turn, C the point where it is struck by the perpendicular blow P. Draw CD perpendicular to AB.

Since the centre of gravity lies in AB and so does not move, the impulses impressed on the body must form a couple. $\{124\ (1)\}$. These impulses are P and the re-actions of AB. Hence the resultant of the latter forces must be equal and opposite to P.

We have yet however to ascertain *where* the resultant reaction of the axis acts, and for this purpose let it be reduced to two forces R, R' at A and B, perpendicular to the plane of the lamina. Then, as we have seen,

$$R + R' + P = 0. \qquad (1)$$

Let ω be the perpendicular velocity generated,

$$BAC = \alpha, \quad AC = 2a, \quad BC = 2b.$$

$$\therefore \ P.CD = \omega \times \text{moment of inertia about } AB,$$

$$= M\omega\left(\frac{a^2}{3}\sin^2\alpha + \frac{b^2}{3}\cos^2\alpha\right) \qquad (2)$$

Now if O the middle point of AB be regarded as origin, and OA the axis of x, then if moments be taken about the axis of y,

$$R.OA - R'.OB - P.OD = \Sigma(mxy)\,\omega.$$

To evaluate $\Sigma(mxy)$ shift the axes through the angle α into parallelism with the sides,

$$\therefore \ x = x'\cos\alpha + y'\sin\alpha,$$

$$y = -x'\sin\alpha + y'\cos\alpha;$$

$$\therefore\ xy = -(x'^2 - y'^2)\sin a\cos a - x'y'(\cos^2 a - \sin^2 a);$$

$$\therefore\ \Sigma(mxy) = -\left\{\Sigma(mx'^2) - \Sigma(my'^2)\right\}\sin a\cos a,$$

$$= -M\left\{\frac{a^2}{3} - \frac{b^2}{3}\right\}\sin a\cos a;$$

$$\therefore\ R.OA - R'.OB - P.OD = -\frac{M}{3}(a^2 - b^2)\,\omega\sin a\cos a. \quad (3)$$

Thus R and R' are fully defined.

This problem may also be solved by means of Poinsot's Central Ellipsoid. If that ellipsoid be constructed, the plane of the impulsive couple producing the motion is conjugate to the diameter AB. When one force therefore (P) of this couple is given, and the distance of its plane ascertained, the other force and the point of AB where it acts are also known.

From either view of the question it appears that no form of the rectangle will cause the impulse on the axis to vanish.

25. The velocity generated in a molecule m will be ωx, and the effective impulse consequently $m\omega x$. The only impressed force is the blow P, if the axis sustains no impulse and therefore exerts no reaction.

Hence
$$P = \Sigma(mx)\,\omega,$$

$$Px' = \Sigma(mx^2)\,\omega,$$

$$Py' = \Sigma(mxy)\omega;$$

$$\therefore\ x' = \frac{\Sigma(mx^2)}{\Sigma(mx)},$$

$$y' = \frac{\Sigma(mxy)}{\Sigma(mx)}.$$

In a lamina the condition of a centre of percussion existing is always satisfied. These simple forms will therefore in such a case suffice to assign its position. They shew the centre of percussion to be the same point as the centre of

pressure of the lamina if it were immersed in a position inclined to the horizon in a liquid whose surface passes through the axis of y.

26. If the formulæ of the preceding question were applied at once to this case, the evaluation of the function would be troublesome in consequence of the discontinuous boundary which the area has when it is referred to rectangular axes. This difficulty is eluded by using polar co-ordinates, with reference to which the boundary becomes continuous.

$$\text{Let } x = r\cos\theta, \quad y = r\sin\theta,$$

$$\begin{aligned}
\therefore\ \Sigma(mxy) &= \Sigma(mr^2\cos\theta\sin\theta), \\
&= \Sigma(r^3\cos\theta\sin\theta\, dr d\theta), \\
&= \tfrac{1}{4}a^4\,\Sigma(\cos\theta\sin\theta\, d\theta), \\
&= \tfrac{1}{8}a^4\sin^2\alpha. \\
\Sigma(mx^2) &= \Sigma(r^3\cos^2\theta\, dr d\theta), \\
&= \tfrac{1}{4}a^4\,\Sigma(\cos^2\theta\, d\theta), \\
&= \tfrac{1}{8}a^4(\alpha + \tfrac{1}{2}\sin 2\alpha). \\
\Sigma(mx) &= \Sigma(r^2\cos\theta\, dr d\theta), \\
&= \tfrac{1}{3}a^3\,\Sigma(\cos\theta\, d\theta), \\
&= \tfrac{1}{3}a^3\sin\alpha. \\
\therefore\ x' &= \tfrac{3}{8}a\,.\,\frac{\alpha + \frac{1}{2}\sin 2\alpha}{\sin\alpha}, \\
&= \tfrac{3}{8}a\left(\frac{\alpha}{\sin\alpha} + \cos\alpha\right), \\
y' &= \tfrac{3}{8}a\sin\alpha.
\end{aligned}$$

27. The centre of gravity begins to move like a free particle under the impressed forces. Its initial motion is perpendicular to the axis, and the blow is not wholly per-

pendicular to the axis, but has a component along the axis. Hence this component requires a reaction of the axis to balance it, and satisfy the conditions of equilibrium which D'Alembert's principle requires.

28. This example is erroneous. For "angular velocity" read "vis viva."

Let l, m, n be the direction cosines of the axis. Then the moment of the couple perpendicular to this axis will be proportional to $Ll + Mm + Nn$, and if ω be the angular velocity generated,

$$(Al^2 + Bm^2 + Cn^2)\,\omega \propto Ll + Mm + Nn,$$

or for brevity,

$$H\omega \propto G \text{ suppose};$$

therefore if ω be a maximum or minimum,

$$0 = \left(\frac{2Al}{H} - \frac{L}{G}\right) dl + \left(\frac{2Bm}{H} - \frac{M}{G}\right) dm + \left(\frac{2Cn}{H} - \frac{N}{G}\right) dn,$$

conditionally with

$$0 = l\,dl + m\,dm + n\,dn;$$

therefore if λ be an arbitrary multiplier,

$$\left.\begin{aligned} 0 &= \frac{2Al}{H} - \frac{L}{G} + \lambda l \\ 0 &= \frac{2Bm}{H} - \frac{M}{G} + \lambda m \\ 0 &= \frac{2Cn}{H} - \frac{N}{G} + \lambda n \end{aligned}\right\} \quad \begin{aligned} &\dots\dots\dots\dots\dots (1) \\ &\dots\dots\dots\dots\dots (2) \\ &\dots\dots\dots\dots\dots (3) \end{aligned}$$

$l\,(1) + m\,(2) + n\,(3)$ gives

$$0 = 1 + \lambda. \quad \therefore \lambda = -1.$$

$$\left.\begin{aligned} \therefore\ \left(\frac{2A}{H} - 1\right) l &= \frac{L}{G} \\ \left(\frac{2B}{H} - 1\right) m &= \frac{M}{G} \\ \left(\frac{2C}{H} - 1\right) n &= \frac{N}{G} \end{aligned}\right\}$$

which define the direction of the axis required.

But if the vis viva generated is to be a maximum, then $\frac{G^2}{H}$ is a maximum;

$$\therefore\ 0 = \left(\frac{Al}{H} - \frac{L}{G}\right) dl + \left(\frac{Bm}{H} - \frac{M}{G}\right) dm + \left(\frac{Cn}{H} - \frac{N}{G}\right) dn\,;$$

under the same conditional relation as before;

$$\therefore\ 0 = \lambda,$$

$$\left.\begin{aligned} \text{and } \frac{Al}{H} &= \frac{L}{G} \\ \frac{Bm}{H} &= \frac{M}{G} \\ \frac{Cn}{H} &= \frac{N}{G} \end{aligned}\right\}$$

which give the equations to the axis to be

$$\frac{Ax}{L} = \frac{By}{M} = \frac{Cz}{N}.$$

This proof like that in Lagrange (*Mec. Analytique*) shews that the vis viva in this case is either a maximum or minimum, without discriminating between these two possible cases. For the considerations which shew that the vis viva is in this case a maximum, vide the *Cambridge Math. Journal*, Vol. IV.

SECTION III.

MOTION OF A BODY ABOUT A FIXED POINT.

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1. THE motion being exhibited by angular velocities about the fixed axes of co-ordinates, we have the velocities of a point x, y, z,

$$\begin{aligned} &z\omega_y - y\omega_z \text{ in direction of } x, \\ &x\omega_z - z\omega_x \;\ldots\ldots\ldots\ldots\ldots\; y, \\ &y\omega_x - x\omega_y \;\ldots\ldots\ldots\ldots\ldots\; z. \end{aligned} \qquad (63)$$

$$\therefore\; v_1^2 = a^2(\omega_z^2 + \omega_y^2),$$

$$v_2^2 = a^2(\omega_x^2 + \omega_z^2),$$

$$v_3^2 = a^2(\omega_x^2 + \omega_y^2).$$

$$v_2^2 + v_3^2 - v_1^2 = a^2\omega_x^2,$$

$$v_1^2 + v_3^2 - v_2^2 = a^2\omega_y^2,$$

$$v_1^2 + v_2^2 - v_3^2 = a^2\omega_z^2;$$

and the equations to the instantaneous axis are

$$\frac{x'}{\sqrt{v_2^2 + v_3^2 - v_1^2}} = \frac{y'}{\sqrt{v_1^2 + v_3^2 - v_2^2}} = \frac{z'}{\sqrt{v_1^2 + v_2^2 - v_3^2}}.$$

2.
$$\begin{aligned} a^2\omega_x^2 = {} & (z\omega_x - y\omega_z)^2 \\ & + (x\omega_z - z\omega_x)^2 \\ & + (y\omega_x - x\omega_y)^2, \end{aligned}$$

is the surface which is the required locus.

3. Since $a\omega_x$ may stand for any constant (c), this example is only a further examination into the nature of the surface just obtained.

Let r be a radius vector of the surface to which l, m, n are direction cosines;

$$\therefore \frac{c^2}{r^2} = \omega_x^2 + \omega_y^2 + \omega_z^2 - (l\omega_x + m\omega_y + n\omega_z)^2.$$

Hence if the direction cosines fulfil the condition

$$l\omega_x + m\omega_y + n\omega_z = 0,$$

r is invariable, or the section circular.

Moreover, if the origin be moved along the line

$$\frac{x}{\omega_x} = \frac{y}{\omega_y} = \frac{z}{\omega_z}$$

to which the circular section is perpendicular, the equation is unaltered in form and the same inference results of an equal circular section parallel to the former. Hence the surface is a circular cylinder with $\dfrac{x}{\omega_x} = \dfrac{y}{\omega_y} = \dfrac{z}{\omega_z}$ for its axis and $c\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ for its radius.

4. Each of these expressions is the square of the body's angular velocity about its instantaneous axis.

5. With the notation of (75),

$$\begin{aligned}\frac{d\theta}{dt} &= a \sin nt \sin\phi + a \cos nt \cos\phi \\ &= a\cos(nt - \phi). \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{d\psi}{dt}\sin\theta &= -a\sin nt\cos\phi + a\cos nt\sin\phi \\ &= -a\sin(nt-\phi). \qquad (2)\end{aligned}$$

$$\frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta = n. \qquad (3)$$

From (2) and (3)

$$\cot\theta = -\frac{n - \dfrac{d\phi}{dt}}{a\sin(nt-\phi)},$$

$$\therefore \cot\theta \cdot \frac{d\theta}{dt} = -\cot(nt-\phi)\cdot\left(n - \frac{d\phi}{dt}\right);$$

$$\therefore \sin\theta . \sin(nt-\phi) = \text{constant}.$$

Now the co-ordinate axes may be so assumed that this constant may be zero, ϕ being 0 when $t = 0$,

$$\therefore nt - \phi = 0 \text{ throughout the motion,}$$

$$\text{or } \phi = nt;$$

$$\therefore \frac{d\theta}{dt} = a;$$

therefore in reference to the figure of (75), the motion of the body is such that C recedes with uniform velocity from Z and the body revolves uniformly about an axis moving with itself and always passing through C.

This and the following are geometrical examples to shew how the nature of the motion may be obtained and described when ω_1, ω_2, ω_3 the elements of the motion are known, and, conversely, how ω_1, ω_2, ω_3 may be assigned when the motion is described.

6. The conditions here given are

$$\frac{d\theta}{dt} \text{ constant} = a \text{ suppose, and } \phi = 0;$$

$$\therefore a = \omega_1 \sin\phi + \omega_2 \cos\phi = \omega_2,$$

$$\frac{d\psi}{dt}\sin\theta = -\omega_1,$$

$$\frac{d\psi}{dt}\cos\theta = \omega_3.$$

$$\theta = at + \beta;$$

$$\therefore \omega_1 = -\frac{d\psi}{dt}\sin(at+\beta),$$

$$\omega_3 = \frac{d\psi}{dt}\cos(at+\beta).$$

7. Let the rod have a constant inclination a to the

vertical line through its fixed end, and an angular velocity ω about the same line. Let $2a$ be the length of the rod.

If m be the mass of an element at a distance x from the fixed end, the effective moving force on this element is $mx\omega^2 \sin\alpha$ towards the vertical line above-mentioned, and the resultant of such forces is $\Sigma(mx)\,\omega^2 \sin\alpha$, or $ma\omega^2 \sin\alpha$, and the moment of this resultant about the fixed end

$$= \frac{M}{2a} \int_x^{0\;\;2a} \omega^2 x^2 \sin\alpha \cos\alpha$$

$$= \tfrac{4}{3} M a^2 \omega^2 \sin\alpha \cos\alpha.$$

$$\therefore\ \tfrac{4}{3} M a^2 \sin\alpha \cos\alpha\, \omega^2 = M a g \sin\alpha\,;$$

$$\cos\alpha = \tfrac{3}{4} \cdot \frac{g}{a\omega^2}.$$

Again, if the reaction of the fixed point, which must be in the vertical plane through the rod because the effective and other impressed force is in this plane, be represented by X, Y horizontally and vertically,

$$\left.\begin{aligned} 0 &= X + M\omega^2 a \sin\alpha, \\ 0 &= Y + Mg. \end{aligned}\right\}$$

which assigns the pressure at the fixed extremity.

If ψ be the angle which the direction of this pressure makes with the vertical,

$$\tan\psi = \frac{X}{Y},$$

$$= \frac{\omega^2 a \sin\alpha}{g},$$

$$= \tfrac{3}{4} \cdot \tan\alpha.$$

8. In this case

$$\frac{d\phi}{dt} + k\left(\frac{\cos^2\phi}{A} + \frac{\sin^2\phi}{B}\right)\cos\theta = \frac{k}{C}\cos\theta,$$

$$\frac{d\theta}{dt} = -k \sin\theta \sin\phi \cos\phi \left(\frac{1}{A} - \frac{1}{B}\right);$$

$$\therefore \ \frac{d\phi}{dt}\left(\frac{1}{B}-\frac{1}{A}\right)\sin\theta\sin\phi\cos\phi$$

$$= \frac{d\theta}{dt}\left\{\frac{1}{C}-\frac{\cos^2\phi}{A}-\frac{\sin^2\phi}{B}\right\}\cos\theta.$$

$$\frac{\cos\theta}{\sin\theta}\cdot\frac{d\theta}{dt} = \left(\frac{1}{B}-\frac{1}{A}\right)\cdot\frac{\sin\phi\cos\phi}{\frac{1}{C}-\frac{1}{A}-\left(\frac{1}{B}-\frac{1}{A}\right)\sin^2\phi}\cdot\frac{d\phi}{dt};$$

$$\therefore \ \log_e\sin\theta = -\tfrac{1}{2}\log_e\left\{\frac{1}{C}-\frac{1}{A}-\left(\frac{1}{B}-\frac{1}{A}\right)\sin^2\phi\right\} + \text{correction};$$

$$\therefore \ \sin^2\theta\left\{\frac{1}{C}-\frac{1}{A}-\left(\frac{1}{B}-\frac{1}{A}\right)\sin^2\phi\right\} = \text{constant};$$

therefore if for any one value of ϕ

$$\frac{1}{C}-\frac{1}{A} = \left(\frac{1}{B}-\frac{1}{A}\right)\sin^2\phi \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

while θ is finite, then the constant = 0, and equation (1) is true throughout the motion.

From it we have

$$\left(\frac{1}{C}-\frac{1}{A}\right)(1+\tan^2\phi) = \left(\frac{1}{B}-\frac{1}{A}\right)\tan^2\phi,$$

$$\left(\frac{1}{C}-\frac{1}{B}\right)\tan^2\phi = \frac{1}{A}-\frac{1}{C}.$$

9. The equations of (78) serve two purposes:

(1) If the forces in action be assigned, ω_1, ω_2, ω_3 can be found if these equations are integrable, and the nature of the motion is thence known. See Ex. (10).

(2) If the motion be given, the forces required to produce it can to a certain degree be determined, of which the present example is an instance. The forces can only be determinate as to some of their conditions and properties. Thus we determine in this example the couples, without assigning the forces which compose them.

The given conditions may be expressed

$$\frac{d\psi}{dt} = \alpha, \quad \frac{d\phi}{dt} = \beta, \quad \theta \text{ constant};$$

$$\therefore\ 0 = \omega_1 \sin\phi + \omega_2 \cos\phi, \tag{75}$$

$$\alpha \sin\theta = -\omega_1 \cos\phi + \omega_2 \sin\phi,$$

$$\beta + \alpha\cos\theta = \omega_3;$$

$$\therefore\ \omega_1 = -\alpha\sin\theta\cos\phi,$$

$$\omega_2 = \alpha\sin\theta\sin\phi.$$

$$\frac{d\omega_1}{dt} = \alpha\beta\sin\theta\sin\phi, \quad \frac{d\omega_2}{dt} = \alpha\beta\sin\theta\cos\phi, \quad \frac{d\omega_3}{dt} = 0;$$

$$\therefore\ L = A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3, \tag{78}$$

$$= \alpha\left\{(A + C - B)\,\beta + (C - B)\,\alpha\cos\theta\right\}\sin\theta\sin\phi,$$

$$\propto \sin\phi.$$

$$M = \alpha\left\{(B + C - A)\,\beta - (A - C)\,\alpha\cos\theta\right\}\sin\theta\cos\phi,$$

$$\propto \cos\phi.$$

$$N = \alpha^2(A - B)\sin^2\theta\sin\phi\cos\phi,$$

$$\propto \sin 2\phi.$$

If $A = B$, the couple in plane $ZC = M\cos\phi + L\sin\phi \propto 1$,

...... perpendicular ... $= -L\cos\phi + M\sin\phi = 0$,

and $N = 0$.

In the second case, $\dfrac{d\theta}{dt} = \alpha$, $\dfrac{d\phi}{dt} = \beta$; therefore by a similar process

$$L = \alpha\beta(A + C - B)\cos\phi,$$

$$M = \alpha\beta(A - C - B)\sin\phi,$$

$$N = -\alpha^2(A - B)\sin\phi\cos\phi.$$

If $A = B$, the couple in $ZC = 0$,

that perpendicular to $ZC \propto 1$,

and $N = 0$.

10. In this example

$$A\frac{d\omega_1}{dt} = (A - C)\,\omega_2\omega_3 + a\sin nt \qquad (1)$$

$$A\frac{d\omega_2}{dt} = (A - C)\,\omega_1\omega_3 + a\cos nt, \qquad (2)$$

and ω_3 is constant, (78); therefore after differentiating (1) and substituting for $\frac{d\omega_2}{dt}$, we have

$$A^2\frac{d^2\omega_1}{dt^2} + (A - C)^2\omega_3^2\omega_1 = \left\{(A - C)\,\omega_3 + An\right\} a\cos nt.$$

$$\frac{d^2\omega_1}{dt^2} + m^2\omega_1 = \frac{m+n}{A}\,a\cos nt,$$

$$\omega_1 = E\cos(mt + F) + \frac{1}{A}\cdot\frac{a}{m-n}\cos nt.$$

Now when $t = 0$, ω_1 and ω_2 vanish,

$$\therefore\ \frac{d\omega_1}{dt} = 0 \text{ also};$$

$$\therefore\ \left.\begin{aligned} 0 &= E\cos F + \frac{1}{A}\cdot\frac{a}{m-n}, \\ 0 &= -E\sin F; \end{aligned}\right\}$$

$$\therefore\ F = 0, \text{ and } E = -\frac{1}{A}\cdot\frac{a}{m-n}.$$

$$\therefore\ \omega_1 = \frac{1}{A}\cdot\frac{a}{m-n}(\cos nt - \cos mt).$$

Again, $m\omega_2 = \frac{d\omega_1}{dt} - \frac{a}{A}\sin nt$;

$$\therefore\ \omega_2 = \frac{1}{A}\cdot\frac{a}{m-n}(\sin mt - \sin nt).$$

If $m = n$, we must evaluate these expressions on the sup-

position of their assuming an indeterminate form as m in the course of its variation assumes the value n;

$$\therefore \left.\begin{aligned} A\omega_1 &= at \sin nt \\ A\omega_2 &= at \cos nt \end{aligned}\right\}.$$

11. A cube is a body for which the central ellipsoid becomes a sphere. Therefore the instantaneous axis about which it is once set in motion remains fixed in space and in the body.

By the symmetry of the figure the effective forces are separately in equilibrium. Therefore the reaction of the fixed point must balance the body's weight.

12. The principal moments of this body about its centre of gravity are equal, so that, as in the former case, it revolves about the original axis permanently. The axis of the solid therefore describes a cone about this axis, and its vertex describes a circle of radius $\frac{3}{4} a \sin a$.

13. When an invariable plane exists, we see from (91) that its position can always be assigned if the state of the body's motion at any instant is given. This example is an instance.

The principal moments of this body are in the ratio of $\frac{1}{8}(a^2 + 4b^2)$ and b^2, while the angular velocities which it has initially about its principal axes are as l, m, n; therefore the equation to the invariable plane is

$$0 = \frac{1}{8}(a^2 4b^2)(lx + my) + b^2 nz. \tag{91}$$

It is to be observed that the axis of z is the initial position of the cone's axis.

14. Let $A > B > C$.

$$A(A - C)\omega_1^2 + B(B - C)\omega_2^2 = k^2 - Ch^2,$$

$$B(B - A)\omega_2^2 + C(C - A)\omega_3^2 = k^2 - Ah^2. \tag{81}$$

The former of these expressions is positive and the latter negative; therefore $\dfrac{k^2}{h^2} > C$ and $< A$.

15. With the notation of (86)

$$A = B = \tfrac{1}{2} C.$$

$$\frac{d\theta}{dt} = 0, \qquad \frac{d\psi}{dt} = \frac{k}{A},$$

so that a normal to the disc revolves about a normal to the invariable plane in time $2\pi \cdot \dfrac{A}{k}$.

Now
$$k^2 = A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2,$$
$$= (A^2 \cos^2 \alpha + 4 A^2 \sin^2 \alpha)\, \omega^2,$$
$$= A^2 (1 + 3 \sin^2 \alpha)\, \omega^2 ;$$

therefore the time of revolution is $\dfrac{2\pi}{\omega \sqrt{1 + 3 \sin^2 \alpha}}$, the unit of time being that involved in the measurement of ω.

If to obtain a better view of the motion we have recourse to Poinsot's method of illustration, let C (fig. 12) be the centre of the disc about which it revolves, CFE the oblate spheroid which is the central ellipsoid in this instance, having its equator CE in the plane of the disc and its axis CF perpendicular to the plane of the disc.

Let CA the axis about which the solid originally revolves meet the surface in P. Draw PY a tangent plane at P. Then a parallel plane through C is the invariable plane, and the motion is exhibited by the spheroid rolling on PY, so that the perpendicular CY is fixed, CP and CF describing cones about this latter line. In this case FCY is θ, remaining as we see invariable, and ψ marks the angular motion of the plane FCY about the fixed line CY.

16. Let GA (fig. 13) be the axis of figure, GB the initial axis of rotation, and suppose the solid in its original position when the angular velocity is impressed upon it. Then

$$AGz = \alpha, \quad AGB = \beta.$$

Then if GA be considered in the plane xz, the equation to the invariable plane if GA were the axis of z' would be

$$O = C \cos\beta \,.\, z' - A \sin\beta \,.\, x'.$$

Therefore the invariable plane makes with the axis of figure the angle $\cot^{-1}\left(\frac{A}{C}\tan\beta\right) = \frac{\pi}{2} - \alpha$, and therefore Gz is perpendicular to it. Therefore the axis of figure of the body describes a circular right cone about Gz. (89).

17. Let CGA (fig. 14) be the central spheroid of the body, PY the tangent plane against which it rolls. Then GY the perpendicular on this tangent plane is fixed in space, and if P be the point of contact at any instant between the plane and the spheroid, GP is the instantaneous axis.

(1) Since GY is invariable, YP is invariable also; therefore P traces on the plane a circle about Y, and GP traces a cone about GY with YGP for its semi-vertical angle.

Now $$\sec PGY = \frac{GP}{GY} = \frac{GP\sqrt{GA^2 + GC^2 - GP^2}}{GA \,.\, GC};$$

therefore $\sec PGY$ and consequently PGY is greatest when

$$2PG^2 = AG^2 + CG^2,$$

or by the nature of the central spheroid when

$$\frac{2}{Q} = \frac{1}{A} + \frac{1}{C},$$

in which case Q is a harmonic mean between A and C.

(2) Since the angle YGC is likewise invariable, GC traces a cone about GY.

Now $GA^2 = GY^2 + (GA^2 - GC^2)\cos^2 CGY$. (Hymers' *Conic Sections*, 135).

But $GY^2(GA^2 + GC^2 - GP^2) = GA^2 \,.\, GC^2$;

$$\therefore \frac{1}{A} = \frac{\frac{1}{AC}}{\frac{1}{A} + \frac{1}{C} - \frac{1}{Q}} + \left(\frac{1}{A} - \frac{1}{C}\right) \cos^2 CGY,$$

$$\frac{\sin^2 CGY}{A} + \frac{\cos^2 CGY}{C} = \frac{Q}{(A+C)Q - AC} (\sin^2 CGY + \cos^2 CGY),$$

$$\sin^2 CGY \left\{\frac{1}{A} - \frac{Q}{(A+C)Q - AC}\right\} = \cos^2 CGY \left\{\frac{Q}{(A+C)Q - AC} - \frac{1}{C}\right\},$$

$$\frac{\sin^2 CGY}{A} . (CQ - AC) = \frac{\cos^2 CGY}{C} (AC - AQ),$$

$$\tan^2 CGY = \frac{A^2}{C^2} . \frac{C - Q}{Q - A}.$$

(3) Since CGP is invariable during the motion, GP will describe a cone relatively to GC, *i. e.* if we suppose the body's axis of figure and the instantaneous axis marked by visible lines, an observer at G unconscious of the body's motion and looking on GC as fixed in space, would say that the instantaneous axis was describing a cone about it.

$$\text{Now} \quad \frac{1}{PG^2} = \frac{\sin^2 PGC}{GA^2} + \frac{\cos^2 PGC}{GC^2},$$

$$Q (\cos^2 PGC + \sin^2 PGC) = A \sin^2 PGC + C \cos^2 PGC,$$

$$\tan^2 PGC = \frac{Q - C}{A - Q}.$$

18. Let x, y, z be the co-ordinates in space of a particle of the body at time t; x', y', z' its invariable co-ordinates referred to axes fixed in the body; ω_1, ω_2, ω_3 angular velocities about lines coinciding with the latter axes which are adequate to exhibit the body's motion.

Then if a_1, $a_2 \ldots b_1 \ldots$ be the direction cosines which exhibit the positions of the latter axes in reference to the former, by the method of (82),

$$y\frac{dz}{dt} - z\frac{dy}{dt} = a_1\left\{(y'^2 + z'^2)\,\omega_1 - x'y'\,\omega_2 - x'z'\,\omega_3\right\}$$
$$+ a_2\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$
$$+ a_3\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

$$z\frac{dx}{dt} - x\frac{dz}{dt} = b_1\{\ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c_1\{\ldots\ldots\ldots\ldots$$

Now $a_1, b_1, c_1 \ldots$ and also $\omega_1, \omega_2, \omega_3$ are in general variable; it cannot therefore be in general possible so to assume x', y', z' that the three expressions above may be separately constant.

19. Since the centre of gravity of the cube does not move, the impulses acting upon it form a couple consisting of the given impulse and an equal contrary impulse at the centre of gravity. The cube is a body for which the central ellipsoid is a sphere. Hence the initial instantaneous axis is perpendicular to the plane of the impulsive couple. Hence if a plane be drawn through the centre and the direction of the given impulse, the initial axis of rotation is a straight line through the centre perpendicular to this plane.

The same result arises from the equations of rotatory motion in (99).

20. Let the cube be referred to axes through its centre of gravity G parallel to its edges, and let each of the blows applied $= P$ in magnitude and act in a direction represented in the figure 15.

Let $\omega_x, \omega_y, \omega_z$ be the initial angular velocities about the co-ordinate axes; then because these are principal axes the angular velocities are in the ratio of

$$-(P_2 + P_3),\quad -(P_1 + P_3),\quad -(P_1 + P_2); \qquad (100)$$

and the equations to the instantaneous axis are

$$\frac{x'}{P_3 + P_2} = \frac{y'}{P_3 + P_1} = \frac{z'}{P_1 + P_2},$$

and thus its position will vary according to the signs of the quantities P_1, P_2, P_3 whose arithmetic values are, we know, equal.

21. The rod describes a cone about a normal to its invariable plane. The invariable plane therefore is to be horizontal. Hence the plane of the original couple is to be horizontal. This couple arises from the blow applied at the opposite reaction of the fixed point. The blow applied must therefore act in the horizontal plane through the centre of gravity.

Such a blow cannot in general be applied to the rod unless the rod be connected rigidly with other points on which the blow acts.

22. This is the same as II. 27.

23. This result is obtained by combining the equations of (101) by addition, after multiplying them in succession,

(1) by $\bar{x}, \bar{y}, \bar{z}$;

(2) by $\omega_1, \omega_2, \omega_3$.

24. Let the body be referred to axes parallel to its sides CA, CB and originating in its centre of gravity G (fig. 16). Let R be the blow at C, ω_x, ω_y the angular velocities instantaneously generated about Gx and Gy;

$$\therefore\ R\,.\,\frac{CB}{3} = \Sigma m\,\{y\,(y\,\omega_x - x\,\omega_y) - z\,(x\,\omega_z - z\,\omega_x)\},$$

$$= \omega_x\,\Sigma\,(my^2) - \omega_y\,\Sigma\,(mxy). \qquad (99)$$

$$-R\,.\,\frac{CA}{3} = \omega_y\,\Sigma\,(mx_2) - \omega_x\,\Sigma\,(mxy).$$

Now $\quad \Sigma\,(mx^2) = \frac{1}{18}\,M\,.\,CA^2,$

$$\Sigma\,(my^2) = \tfrac{1}{18}\,M\,.\,CB^2.$$

If x', y' denote co-ordinates parallel to those already used but originating in C,

$$\Sigma(mxy) = \Sigma m\left(x' - \frac{CA}{3}\right)\left(y' - \frac{CB}{3}\right),$$

$$= \tfrac{1}{12}M.CA.CB - \tfrac{1}{9}M.CA^2 - \tfrac{1}{9}M.CB^2 + \tfrac{1}{9}M.CA.CB,$$

$$= (\tfrac{7}{36}CA.CB - \tfrac{1}{9}CA^2 - \tfrac{1}{9}CB^2)M; \qquad \text{(I. i. } \gamma.)$$

$$\therefore \frac{CA}{CB} = \frac{\omega_x \Sigma(mxy) - \omega_y \Sigma(mx^2)}{\omega_x \Sigma(my^2) - \omega_y \Sigma(mxy)}.$$

$$CA\left\{\tfrac{1}{18}CB^2.\omega_x - (\tfrac{7}{36}CA.CB - \tfrac{1}{9}CA^2 - \tfrac{1}{9}CB^2)\omega_y\right\}$$

$$= CB\left\{(\tfrac{7}{36}CA.CB - \tfrac{1}{9}CA^2 - \tfrac{1}{9}CB^2)\omega_x - \tfrac{1}{18}CA^2.\omega_y\right\},$$

$$\omega_x\left\{\tfrac{1}{9}(CA^2 + CB^2).CB - \tfrac{5}{36}CA.CB^2\right\} \qquad \text{(A)}$$

$$= -\omega_y\left\{\tfrac{1}{9}(CA^2 + CB^2)CA - \tfrac{5}{36}CA^2.CB\right\};$$

$$\therefore \omega_x.CB + \omega_y.CA = 0,$$

and the equation to the initial instantaneous axis is

$$\frac{x'}{CA} + \frac{y'}{CB} = 0,$$

a straight line parallel to AB.

This result has arisen from equating one factor of (A) to zero, but the other is always finite.

For if $4(CA^2 + CB^2) = 5CA.CB$,

$$4(CA - CB)^2 + 3CA.CB = 0,$$

which cannot be satisfied.

25. Let the angular velocities be changed by the check to ω_x', ω_y', ω_z', and let the reactions of the axis be expressed by

$$\left.\begin{matrix}F\\G\\H\end{matrix}\right\} \text{ parallel to } \left\{\begin{matrix}x\\y\\z\end{matrix}\right. \text{ at the point } \left\{\begin{matrix}la\\ma\\na,\end{matrix}\right.$$

$$\left.\begin{matrix}F'\\G'\\H'\end{matrix}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \left\{\begin{matrix}-la\\-ma\\-na.\end{matrix}\right.$$

$$\therefore\ A(\omega_x' - \omega_x) - C'(\omega_y' - \omega_y) - B'(\omega_z' - \omega_z)$$
$$= (H - H')\,ma - (G - G')\,na, \quad (1)$$

$$B(\omega_y' - \omega_y) - C'(\omega_x' - \omega_x) - A'(\omega_z' - \omega_z)$$
$$= (F - F')\,na - (H - H')\,la, \quad (2)$$

$$C(\omega_z' - \omega_z) - B'(\omega_x' - \omega_x) - A'(\omega_y' - \omega_y)$$
$$= (G - G')\,la - (F - F')\,ma, \quad (3)$$

according to the notation of (99).

Again the new motion is a rotation about the axis whose direction cosines are l, m, n;

$$\therefore\ \frac{\omega_x'}{l} = \frac{\omega_y'}{m} = \frac{\omega_z'}{n}. \quad (4)\quad (5)$$

Also each of the two resultant impulses is perpendicular to the axis;

$$\therefore\ Fl + Gm + Hn = 0,$$
$$F'l + G'm + H'n = 0;$$
$$\text{or}\ (F - F')\,l + (G - G')\,m + (H - H')\,n = 0. \quad (6)$$

There are thus six equations for determining ω_x', ω_y', ω_z', and thereby the new angular velocity, and also the differences $F - F'$, $G - G'$, $H - H'$. And these differences of the impulses are all that can be expected to be determined, while the particular value of each is not assigned, for if the axis were struck by parallel blows at points on opposite sides of the origin and equidistant from it, these would have no effect on the motion.

26. Since the axes of the figure are principal axes, if R be the blow applied at x', y' perpendicular to the plane,

$$A\omega_x = -R\,.\,y',$$
$$B\omega_y = Rx'. \quad (100)$$
$$\therefore\ 0 = Ax'\omega_x + By'\omega_y,$$
$$0 = b^2x'\,.\,\omega_x + a^2y'\,.\,\omega_y;$$

therefore the equation to the instantaneous axis is

$$0 = b^2 x' . x + a^2 y' . y,$$

and this is identical with

$$0 = ay - bx;$$

$$\therefore \ ay' + bx' = 0,$$

or x', y' is a point in the line $ay + bx = 0$.

The central ellipsoid in the subsequent motion will roll against a plane perpendicular to the lamina. The point of contact will therefore change. The instantaneous axis will therefore alter its position and the angular velocity about it will vary.

27. Let the blow be applied at the point α, β, γ of the surface, and let l, m, n be the direction cosines of its direction.

The axes of figure being principal axes,

$$\left.\begin{aligned} A\omega_x &= -(n\beta - m\gamma) R \\ B\omega_y &= -(l\gamma - n\alpha) R \\ C\omega_z &= -(m\alpha - l\beta) R. \end{aligned}\right\} \qquad (100)$$

Since the blow is normal at α, β, γ,

$$\frac{la^2}{\alpha} = \frac{mb^2}{\beta} = \frac{nc^2}{\gamma},$$

and from the equation to the conical surface in which the instantaneous axis lies,

$$\frac{\omega_x^{\ 2}}{b^2 - c^2} + \frac{\omega_y^{\ 2}}{c^2 - a^2} + \frac{\omega_z^{\ 2}}{a^2 - b^2} = 0;$$

$$\therefore \ 0 = \frac{\left(\frac{1}{c^2} - \frac{1}{b^2}\right)^2 \beta^2\gamma^2}{A^2 (b^2 - c^2)^2} + \frac{\left(\frac{1}{a^2} - \frac{1}{c^2}\right)^2 \alpha^2\gamma^2}{B^2 (a^2 - c^2)^2} + \frac{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 \alpha^2\beta^2}{C^2 (a^2 - b^2)^2};$$

$$\therefore \ 0 = \frac{b^2 - c^2}{(b^2 + c^2)^2} \cdot \frac{a^4}{\alpha^2} + \frac{c^2 - a^2}{(c^2 + a^2)^2} \cdot \frac{b^4}{\beta^2} + \frac{a^2 - b^2}{(a^2 + b^2)^2} \cdot \frac{c^4}{\gamma^2}.$$

Hence the point α, β, γ lies on the intersection of the ellipsoid and the cone whose equation is

$$0 = \frac{b^2 - c^2}{(b^2 + c^2)^2} \cdot \frac{a^4}{x^2} + \frac{c^2 - a^2}{(c^2 + a^2)^2} \cdot \frac{b^4}{y^2} + \frac{a^2 - b^2}{(a^2 + b^2)^2} \cdot \frac{c^4}{z^2}.$$

28. From the equations obtained in the preceding question, the instantaneous axis has for its equations

$$\frac{A}{n\beta - m\gamma}x' = \frac{B}{l\gamma - n\alpha}y' = \frac{C}{m\alpha - l\beta}z',$$

$$\text{or } \frac{b^2 + c^2}{\frac{1}{b^2} - \frac{1}{c^2}} \cdot \frac{x'}{\beta\gamma} = \frac{a^2 + c^2}{\frac{1}{a^2} - \frac{1}{c^2}} \cdot \frac{y'}{\alpha\gamma} = \frac{a^2 + b^2}{\frac{1}{b^2} - \frac{1}{a^2}} \cdot \frac{z'}{\alpha\beta},$$

$$\text{or } \frac{b^2 + c^2}{b^2 - c^2} \cdot \frac{x'\alpha}{a^2} = \frac{a^2 + c^2}{c^2 - a^2} \cdot \frac{y'\beta}{b^2} = \frac{a^2 + b^2}{a^2 - b^2} \cdot \frac{z'\gamma}{c^2}.$$

Hence the angular velocities of the body about the co-ordinate axes are proportional to

$$\frac{a^2}{\alpha} \cdot \frac{b^2 - c^2}{b^2 + c^2}, \quad \frac{b^2}{\beta} \cdot \frac{c^2 - a^2}{c^2 + a^2}, \quad \frac{c^2}{\gamma} \cdot \frac{a^2 - b^2}{a^2 + b^2};$$

and since the co-ordinate axes at the instant considered coincide with principal axes of the body, hence (91) the equation to the invariable plane is

$$0 = A \cdot \frac{a^2}{\alpha} \cdot \frac{b^2 - c^2}{b^2 + c^2}x + B \cdot \frac{b^2}{\beta} \cdot \frac{c^2 - a^2}{c^2 + a^2}y + C \cdot \frac{c^2}{\gamma} \cdot \frac{a^2 - b^2}{a^2 + b^2}z,$$

$$\text{or } 0 = a^2(b^2 - c^2)\frac{x}{\alpha} + b^2(c^2 - a^2)\frac{y}{\beta} + c^2(a^2 - b^2)\frac{z}{\gamma}.$$

SECTION IV.

MOTION OF A FREE RIGID BODY.

1. The statement in this example is not true, as the following investigation shews.

If the motion of the body be exhibited by the velocities u_1, v_1, w_1 of the point which is coincident with the origin at the moment considered, and by angular velocities ω_x, ω_y, ω_z about the co-ordinate axes, the result in (103) supplies six equations from the given component velocities of the two points, and if these were independent, they would suffice to determine the six elements by which the body's motion is to be exhibited.

The data will however be insufficient if the equations

$$u = u_1 + z\omega_y - y\omega_z,$$

$$v = v_1 + x\omega_z - z\omega_x,$$

$$w = w_1 + y\omega_x - x\omega_y,$$

$$u' = u_1 + z'\omega_y - y'\omega_z,$$

$$v' = v_1 + x'\omega_z - z'\omega_x,$$

$$w' = w_1 + y'\omega_x - x'\omega_y,$$

are not independent.

Now these equations give

$$u - u' = (z - z')\omega_y - (y - y')\omega_z, \tag{1}$$

$$v - v' = (x - x')\omega_z - (z - z')\omega_x, \tag{2}$$

$$w - w' = (y - y')\omega_x - (x - x')\omega_y. \tag{3}$$

$(x - x') \times (1) + (y - y') \times (2)$ produces (3) if

$$0 = (x - x')(u - u') + (y - y')(v - v') + (z - z')(w - w'),$$

and unless this condition is satisfied the equations are incompatible.

Hence more is required to be known than the motion of two points in order to assign completely the elements of a rigid body's motion.

2. Let P (fig. 17) be the point whose co-ordinates are

$$OM = a, \quad MN = \beta, \quad NP = \gamma.$$

Then the motion consists of three rotations $l\omega$, $m\omega$, $n\omega$ about lines parallel to the co-ordinate axes originating in P.

The motion will not be altered if equal and opposite angular velocities $n\omega$ be impressed about the axis of z and also about a parallel line through M. Then the five angular velocities $n\omega$ are reducible to

(1) An angular velocity $n\omega$ about Oz,

(2) A couple of rotatory motion where the angular velocity is $n\omega$ and the arm MN, and this is equivalent to a linear velocity $n\beta\omega$ parallel to x.

(3) A couple of angular velocity $n\omega$ and arm OM, equivalent to a linear velocity $-na\omega$ parallel to y.

If the other component velocities be similarly transformed, they give at last

angular velocities $l\omega$, $m\omega$, $n\omega$ about the co-ordinate axes,

linear .... $(n\beta - m\gamma)\,\omega$, $(l\gamma - na)\,\omega$, $(ma - l\beta)\,\omega$

parallel to x, y, z.

3. Let C (fig. 18) be the centre of the disc, A the point about which its centre describes the circle. In AC take a point O so that $AO\,.\,\Omega = CO\,.\,\omega$. Then if O be a point actually or hypothetically united with C, it will have no velocity. Also if the disc were revolving about an axis through O perpendicular to its plane and had about this an angular velocity ω', then the velocities of any point P of the disc would be

$$\left.\begin{array}{l} \omega'\,.\,OM \text{ perpendicular to } AC \\ -\,\omega'\,.\,PM \text{ in direction } AC, \end{array}\right\}$$

PM being perpendicular to AC or AC produced.

But actually P has the velocities

$$\Omega . AC + \omega . CM = (\Omega + \omega)\, OM \text{ perpendicular to } AC,$$

and $-\Omega . PM - \omega . PM = -(\Omega + \omega)\, PM$ in direction of AC; therefore if $\omega' = \Omega + \omega$, a single rotation about the axis through O can exhibit the motion. Since $AO = \dfrac{a\omega}{\omega + \Omega}$, the locus of the instantaneous axis is a cylinder.

If $\omega + \Omega = 0$, then the velocity of the point M is

$$\Omega . AM + \omega . CM \text{ perpendicular to } AC = \Omega . AC,$$

and this is true for every point in the diameter through A. Hence every point in this diameter and consequently the whole disc has the same velocity as that of C and in the same direction. The motion is therefore a translation.

Obs. In the general case why is not $\Omega . AC$ the velocity of translation of the disc, and consequently $\Omega . AC + \omega . CM$ the velocity of P perpendicular to AC?

4. Let the ecliptic be plane of xy. Then the condition of reduction to a single rotation is

$$0 = u . \omega_x + v . \omega_y.$$

Now whenever the Earth's centre returns to the same point of the ecliptic u and v assume the same values, but ω_x and ω_y change from year to year in consequence of Precession. Hence if the preceding equation be satisfied at a certain point of the orbit in any one year, it will not be generally satisfied again, or the motion will not generally be one of rotation about a single axis.

5. Let $A, B, C, \ldots$ (fig. 19) be the traces of the axes on a plane perpendicular to them all, $\omega_1, \omega_2 \ldots$ the angular velocities of the body about them respectively.

Join AB, and take $\omega_1 . Aa = \omega_2 . Ba$. Then ω_1 and ω_2 may be replaced by a single rotation $\omega_1 + \omega_2$ about an axis through a.

Join Ca and take $(\omega_1 + \omega_2).ab = \omega_3.Cb$. Then $\omega_1, \omega_2, \omega_3$ may be replaced by $\omega_1 + \omega_2 + \omega_3$ about an axis through b.

By following this process we obtain at last the position of a single axis about which a velocity of rotation equal to the sum of the given velocities will exhibit the motion. The construction is similar to that of finding the centre of gravity. Hence the axis may be defined by its passing through the centre of gravity of a series of particles at A, B, C,... whose masses are proportional to $\omega_1, \omega_2, \omega_3$.

If the points A, B,...be exhibited by rectangular co-ordinates in their plane, the co-ordinates of the trace of the resulting axis are

$$\bar{x} = \frac{\Sigma(\omega x)}{\Sigma(\omega)}, \quad \bar{y} = \frac{\Sigma(\omega y)}{\Sigma(\omega)}.$$

If $\Sigma(\omega) = 0$, the motion is not reducible to a single rotation.

6. Let the lines be referred to co-ordinate axes so that the axis of z is in direction of the shortest distance between the lines and the plane of xz bisects the angle $(2a)$ between the lines. Then a rotation ω about one consists of

$$\left.\begin{matrix}\omega\cos a\\ \omega\sin a\end{matrix}\right\}\text{ about an axis parallel to axis of }\left\{\begin{matrix}x,\\ y,\end{matrix}\right. \qquad (72)$$

and ω' about the other consists of

$$\left.\begin{matrix}\omega'\cos a\\ -\omega'\sin a\end{matrix}\right\}\text{ about an axis parallel to axis of }\left\{\begin{matrix}x,\\ y.\end{matrix}\right.$$

Hence

(1) If $\omega = \omega'$, the motion is a rotation about the axis of x and a translation in direction of the axis of x,

(2) If $\omega + \omega' = 0$, the motion is rotation about y combined with translation along y.

In either case one of the co-ordinate axes is the central axis and is equally inclined to each of the given axes.

7. Let a = radius of the wheel, M its mass and Mk^2 its moment of inertia, m the mass of the suspended weight. At

time t let x be the distance of the suspended weight from a given horizontal plane, $\bar{x}$ that of the wheel's centre, θ the angle through which the wheel has moved from a given position, T the tension of the string, R the reaction of the point to which the wheel's centre is attached. Then (1) the equations of motion of the wheel's centre are constructed like those of a particle;

$$\therefore\ M\frac{d^2\bar{x}}{dt^2} = Mg + T - R, \tag{1}$$

and the equations of its rotation are constructed like those for rotation about a fixed axis,

$$\therefore\ Mk^2\frac{d^2\theta}{dt^2} = Ta. \tag{2}$$

(2) The equation for the motion of m is

$$m\frac{d^2x}{dt^2} = mg - T. \tag{3}$$

Also from the string being inextensible

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt} + a\frac{d\theta}{dt}. \tag{4}$$

The solution now branches off according to the condition selected from the two before us.

I. If the wheel is fixed,

$$\bar{x} \text{ is constant,}$$

$$\frac{d\bar{x}}{dt} \text{ and } \frac{d^2\bar{x}}{dt^2} \text{ are zero.}$$

$$\therefore\ mg = m\frac{d^2x}{dt^2} + \frac{Mk^2}{a}\cdot\frac{d^2\theta}{dt} \text{ from (2) (3)}$$

$$= \left(m + M\frac{k^2}{a^2}\right)\frac{d^2x}{dt^2},$$

or the accelerating force on the descending weight is uniform and

$$= \frac{mg}{m + M\dfrac{k^2}{a^2}},$$

and the angular accelerating force on the wheel is

$$\frac{m}{m + M\dfrac{k^2}{a^2}} \cdot \frac{g}{a}.$$

II. If the wheel is free, $R = 0$;

$$\therefore \; 0 = \frac{d^2\bar{x}}{dt^2} + a\frac{d^2\theta}{dt^2} - \frac{d^2x}{dt^2},$$

$$= g + \frac{T}{M} + \frac{Ta^2}{Mk^2} - g + \frac{T}{m};$$

$$\therefore \; T = 0.$$

Therefore the wheel does not revolve, and the whole descends under the uniform accelerating force g like a solid mass.

The comparison of these results is intended to elucidate the remark in (119).

8. Let C (fig. 20) be the centre of the wheel moving along the straight line OC with a velocity v, ω the angular velocity of the wheel about an axis through C.

Since the whole motion of the point C is in OC, the trace of the instantaneous axis must be in the straight line perpendicular to OC. Let it be P.

$$\text{Then } v = \omega \, . \, CP;$$

$$\therefore \; \frac{1}{CP} \propto \omega \propto \text{time from } O.$$

$$\text{But } OC \propto \text{time};$$

$$\therefore \; OC \, . \, CP \propto 1.$$

Therefore the locus of P is a rectangular hyperbola of which O is the centre and OC one of the asymptotes. The locus of the instantaneous axis is the corresponding hyperbolic cylinder.

9. Since no forces act upon the body the centre of gravity remains at rest if, as is supposed, it is originally at rest. Also the motion about the centre of gravity is like that which would arise if that point were fixed, and therefore the axis, because it is a principal axis, will continue the permanent axis of rotation and the angular velocity will be unaltered.

10. The centre of gravity of the solid moves in a straight line parallel to the direction of the force, and the body oscillates about the centre of gravity as if this centre were fixed.

Hence if at time t the radius through the centre of gravity makes an angle θ with the direction of the force,

$$k^2 \frac{d^2\theta}{dt^2} = -\tfrac{3}{8} af \sin\theta;$$

therefore the length of the simple equivalent pendulum

$$= \frac{k^2}{\frac{3}{8}af} \cdot g,$$

$$= \frac{\frac{2}{5} - (\frac{3}{8})^2}{\frac{3}{8}} \frac{ag}{f},$$

$$= \frac{83}{120} \frac{ag}{f};$$

and the time of a small oscillation is

$$\pi \sqrt{\frac{83}{120} \cdot \frac{a}{f}}.$$

The unit of time is that supposed in the numerical measurement of the force f.

11. If the cube begins to rotate about a diagonal, whereby the centre of gravity is initially at rest, the impulses must form a couple. Since the central ellipsoid in the case of a cube is a sphere, the couple's plane is perpendicular to the proposed diagonal.

If the cube begins to rotate about an edge, the motion is reducible to a translation of the centre of gravity perpendicular to the diagonal plane through that edge, and a rotation about an axis through the centre parallel to the edge. The former will be produced by a force in its direction, the latter by a couple perpendicular to the given edge. The force and couple are also connected as to magnitude by the relation between the velocities of translation and rotation aforesaid.

12. Let R be the blow, ω_x, ω_y the angular velocities which it instantaneously produces about the axes of x and y, M the mass of the body.

Since $M\dfrac{b^2}{12}$ and $M\dfrac{a^2}{12}$ are the moments of inertia of the lamina about the co-ordinate axes, we have

$$M\frac{b^2}{12}\omega_x = -Rk,$$

$$M\frac{a^2}{12}\omega_y = \quad Rh. \quad (100).$$

Hence the equation to the instantaneous axis,

$$\frac{x}{\omega_x} = \frac{y}{\omega_y},$$

becomes $\dfrac{b^2x}{k} + \dfrac{a^2y}{h} = 0.$

But if the centre be free we have besides a velocity $\dfrac{R}{M}$ produced perpendicular to the plane. Hence the initial velocity of a point x, y, is

$$y\omega_x - x\omega_y - \frac{R}{M},$$

and the initial axis is

$$0 = \frac{12\,ky}{b^2} + \frac{12\,hx}{a^2} + 1,$$

a straight line parallel to the former, the perpendicular from the origin on it being

$$= \frac{1}{12}\left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)^{-\frac{1}{2}}.$$

13. First this spheroid must have been struck by a force at the centre in the line along which the centre moves, and secondly by a couple adequate to produce the motion described if the centre were fixed. Now on the latter supposition the axis of the circular cone is normal to the invariable plane and so also to the couple's plane. Thus the plane of the couple is assigned.

The forces therefore must be reducible to the force and couple thus described.

14. Let x be the distance of the new axis from the original one, R the blow, M the mass of the body.

Since R produces in the centre of gravity the linear velocity $\omega' x$,

$$\therefore\ M\omega' x = R,$$

and since it changes the angular velocity about the centre of gravity from ω to ω',

$$\therefore\ \frac{Ma^2}{4}(\omega' - \omega) = Ra.$$

$$\therefore\ a = \frac{a^2}{4x} \cdot \frac{\omega' - \omega}{\omega'},$$

$$x = \frac{a}{4} \cdot \frac{\omega' - \omega}{\omega'}.$$

15. Let x be the distance of the point of impact from the centre of rod (fig. 21). Then if R be the blow, M the rod's mass, this blow will generate a velocity of transla-

tion $\frac{R}{M}$ and a velocity of rotation about the centre of gravity $= \frac{3Rx}{Ma^2}$. In order therefore that A may be at rest

$$\frac{R}{M} = \frac{3Rxa}{Ma^2}, \text{ or } x = \frac{a}{3}.$$

The centre of gravity G now moves along Ox with the uniform velocity v generated in it, and the rod revolves about G with the angular velocity $\frac{v}{a}$. Hence when $OG = vt$, the rod will have moved through the angle

$$\frac{v}{a} . t = \frac{\pi}{2} - AGO;$$

therefore the position of A is defined by

$$x = vt - a \sin \frac{vt}{a}, \quad y = a \cos \frac{vt}{a}.$$

$$\therefore \ x = a \cos^{-1} \frac{y}{a} - \sqrt{a^2 - y^2}.$$

16. Let $2a$ be an edge of the cube, M its mass, P_1, P_2, P_3 the blows equal in magnitude and acting in the lines represented in figure 15. The centre of gravity of the cube being origin, and the co-ordinate axes parallel to the edges of the cube, the equations which give the motion of the centre of gravity are $Mu = P_1$, $Mv = P_2$, $Mw = P_3$; and those which give the motion of rotation are

$$A\omega_1 = -(P_2 + P_3)\, a,$$

$$A\omega_2 = -(P_1 + P_3)\, a,$$

$$A\omega_3 = -(P_1 + P_2)\, a. \quad (124).$$

Hence the equations which are to give the instantaneous axis are

$$0 = \frac{P_1}{M} - \frac{P_1 + P_3}{A} az + \frac{P_1 + P_2}{A} ay,$$

$$0 = \frac{P_2}{M} - \frac{P_1 + P_2}{A} ax + \frac{P_2 + P_3}{A} az,$$

$$0 = \frac{P_3}{M} - \frac{P_2 + P_3}{A} ay + \frac{P_1 + P_3}{A} ax.$$

If these equations are multiplied by $P_2 + P_3$, $P_1 + P_3$, $P_1 + P_2$ in order, and added together,

$$0 = P_2 P_3 + P_1 P_3 + P_1 P_2.$$

Since P_1, P_2, P_3 are equal in magnitude, this equation cannot be satisfied, whether the impulses act as the figure represents them, or the direction of any of them be reversed. Hence there is no single instantaneous axis at a finite distance from the body.

SECTION V.

MISCELLANEOUS EXAMPLES.

1. LET Ox (fig. 22) along which the disc rolls be made axis of x, P any point of the disc at time t when C is the position of the centre of the disc, and Q the point where it touches the plane, x, y, co-ordinates of P, $QCP = \theta$;

$$\therefore \left.\begin{aligned} x &= OQ + r\sin\theta \\ y &= \;\; a \;\; - r\cos\theta \end{aligned}\right\},$$

$$\frac{dx}{dt} = V + r\cos\theta \, . \, \frac{d\theta}{dt},$$

$$= V + r\cos\theta \, . \, \frac{V}{a}, \quad (132).$$

$$\frac{d^2x}{dt^2} = -r\sin\theta \, \frac{V^2}{a^2}.$$

$$\text{Also } \frac{dy}{dt} = r\sin\theta \, . \, \frac{V}{a},$$

$$\frac{d^2y}{dt^2} = r\cos\theta \, . \, \frac{V^2}{a^2}.$$

If then PN be drawn perpendicular to CQ, it appears that the effective forces on P in directions PN, QC are proportional to PN, NC. Hence their resultant is in direction PC and equal to $\frac{V^2}{a^2} \, . \, r$.

2. Let x be the distance of the sphere's centre from a fixed vertical line in the plane of motion, x_1 that of a fixed point of the board, F the horizontal action between the sphere and board, θ the angle through which the sphere has turned from a fixed position, m the mass of the sphere, M that of the board, a the radius of the sphere;

$$\left.\begin{aligned} \therefore\ m\frac{d^2x}{dt^2} &= F \\ mk^2\frac{d^2\theta}{dt^2} &= -Fa \\ M\frac{d^2x_1}{dt^2} &= -F \end{aligned}\right\}.$$

Now the condition of rolling requires that the point of the sphere and board which are in contact have the same velocity, (132);

$$\therefore\ \frac{dx_1}{dt} = \frac{dx}{dt} - a\frac{d\theta}{dt};$$

$$\therefore\ -\frac{F}{M} = \frac{F}{m} + \frac{a^2}{mk^2}F;$$

therefore $F = 0$, and each part of the motion is uniform.

3. Let O (fig. 23) the centre of the fixed sphere be origin of horizontal and vertical co-ordinates of x and y, C the centre of the rolling sphere, θ the angle in space through which it has turned in time t, ϕ the angle through which the other sphere has turned, P the action on the rolling sphere in OC, F that perpendicular to OC and upwards, R, r the radii of the spheres whose centres are O and C, M, m their masses;

$$\begin{aligned} \therefore\ m\frac{d^2x}{dt^2} &= -F\cdot\frac{y}{R+r} + P\frac{x}{R+r}, \\ m\frac{d^2y}{dt^2} &= F\frac{x}{R+r} + P\cdot\frac{y}{R+r} - mg, \\ mk^2\frac{d^2\theta}{dt^2} &= Fr, \\ MK^2\frac{d^2\phi}{dt^2} &= FR. \end{aligned}$$

By condition of rolling the points of the spheres which are in contact must have the same velocity in and perpendicular to OC, (132);

$$\therefore \frac{dx}{dt} \cdot \frac{x}{R+r} + \frac{dy}{dt} \cdot \frac{y}{R+r} = 0,$$

$$\frac{dx}{dt} \cdot \frac{y}{R+r} - \frac{dy}{dt} \cdot \frac{x}{R+r} - r\frac{d\theta}{dt} = R\frac{d\phi}{dt},$$

so that six equations are obtained for determining x, y, θ, ϕ, F, P, in terms of t, if the integrations can be effected.

If the last equation be differentiated,

$$y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = (R+r)\left(r\frac{d^2\theta}{dt^2} + R \cdot \frac{d^2\phi}{dt^2}\right),$$

$$-\frac{F}{m}(R+r) = (R+r)\left(\frac{Fr^2}{mk^2} + \frac{FR^2}{MK^2}\right) - gx,$$

$$gx = F\left\{\frac{R+r}{m} + (R+r)\left(\frac{r^2}{mk^2} + \frac{R^2}{MK^2}\right)\right\},$$

$$F = \frac{1}{R+r} \cdot \frac{gx}{\frac{1}{m} + \frac{r^2}{mk^2} + \frac{R^2}{MK^2}}$$

$$= \frac{1}{R+r} \cdot \frac{gx}{\frac{1}{m} + \frac{5}{2m} + \frac{5}{2M}}$$

$$= \frac{2g}{\frac{7}{m} + \frac{5}{M}} \cdot \sin COy.$$

Hence $\dfrac{d^2\phi}{dt^2}$ and $\dfrac{d^2\theta}{dt^2}$ are at once known.

4. With the notation of question 2, page (61), we have now

$$m\frac{d^2x}{dt^2} = F,$$

$$mk^2\frac{d^2\theta}{dt^2} = -Fa,$$

$$\frac{dx_1}{dt} = \frac{dx}{dt} - a\frac{d\theta}{dt}.$$

But the board is in this case constrained to move with uniformly increasing velocity, so that $\frac{d^2x_1}{dt^2} = c$ a constant;

$$\therefore\ c = \frac{F}{m} + \frac{Fa^2}{mk^2}$$

$$= \frac{F}{m}(1 + \tfrac{5}{2})$$

$$= \tfrac{7}{2}\frac{F}{m},$$

$$F = \tfrac{2}{7}mc.$$

Let T be the horizontal mutual impulsive action when the motion is checked; v, ω the previous velocities of translation and rotation of the sphere which are instantaneously changed to v', ω';

$$\therefore\ \left.\begin{aligned} m(v - v') &= T \\ mk^2(\omega' - \omega) &= Ta \end{aligned}\right\}.$$

Now if V was the velocity of the plane when it was stopped, $v - V = a\omega$, because the previous motion of the sphere was one of rolling, and if the supposition be made that its motion continues of this kind, $v' = a\omega'$;

$$\therefore\ mV = T\left(1 + \frac{a^2}{k^2}\right) = \tfrac{7}{2}T,$$

$$v - v' = \tfrac{2}{7}V.$$

This is the change in the velocity of the sphere.

The horizontal impulse applied to the plane must be equal to its momentum together with T or $\frac{2}{7}mV$. If the impulse which brings the plane to rest is different from this the rolling motion of the sphere is disturbed, although if the problem be pursued further, friction may restore the rolling motion after an interval.

5. Before we introduce the principle of *vis viva* in solving this problem, it may be useful to shew that when gravity is the only force in action and entitled to be taken into account in

forming the function $\Sigma m \int (Xdx + Ydy + Zdz)$, then the value of that function between two positions of the system is the weight of the system multiplied into the vertical space through which its centre of gravity has descended.

For if z be measured vertically, in such case $X = 0$, $Y = 0$, $Z = g$; and if z, z' be the vertical co-ordinates of m in the two positions of the system, $\bar{z}$, $\bar{z}'$ those of the centre of gravity of the system,

$$\Sigma \left\{m \int (Xdx + Ydy + Zdz)\right\} = \Sigma \{mg (z' - z)\}$$
$$= g \Sigma (mz') - g \Sigma (mz)$$
$$= Mg (\bar{z}' - \bar{z}).$$

Now Mg is the weight of the system, and $\bar{z}' - \bar{z}$ the space through which its centre of gravity has descended.

To proceed with the problem before us let C (fig. 24) be the trace of the axis of the cylinder, Q that of the line of contact, P a point of the cylinder in the plane of the paper, $QM = x$, $MP = y$ its co-ordinates described in the enunciation, $OM = \eta$, $MP = \xi$ its parallel co-ordinates referred to an origin fixed in space, θ the inclination of PC to QC produced. Then the effective forces on P are $\dfrac{d^2\eta}{dt^2}$ and $\dfrac{d^2\xi}{dt^2}$ in direction of the co-ordinates.

By the principle of *vis viva* and the nature of rolling motion,

$$\left(\frac{d \,.\, OQ}{dt}\right)^2 + \frac{k^2}{a^2} \cdot \left(\frac{d \,.\, OQ}{dt}\right)^2 = 2g \sin 30^0 \,.\, OQ$$
$$= g \,.\, OQ,$$

$$\text{or } \tfrac{3}{2} \left(\frac{d \,.\, OQ}{dt}\right)^2 = g \,.\, OQ;$$

$$\therefore\ 3 \frac{d^2 \,.\, OQ}{dt^2} = g;$$

$$\therefore\ 3 \frac{d^2\theta}{dt^2} = \frac{g}{a},$$

$$\text{and } \tfrac{3}{2} \left(\frac{d\theta}{dt}\right)^2 = \frac{g \,.\, OQ}{a^2}.$$

$$\text{Now } \eta = OQ + x$$
$$= OQ + CP \sin\theta;$$
$$\therefore \frac{d\eta}{dt} = \frac{d \,.\, OQ}{dt} + CP \,.\, \cos\theta \frac{d\theta}{dt},$$
$$\frac{d^2\eta}{dt} = \frac{g}{3} + CP \,.\, \cos\theta \,.\, \frac{g}{3a} - CP \,.\, \sin\theta \,.\, \frac{2g \,.\, OQ}{3a^2}$$
$$= \frac{g}{3} + (y - a)\frac{g}{3a} - \frac{2gl}{3a^2} x$$
$$= \frac{gy}{3a} - \frac{2glx}{3a^2};$$
$$\text{and } \xi = a + CP \,.\, \cos\theta,$$
$$\therefore \frac{d\xi}{dt} = -CP \sin\theta \frac{d\theta}{dt},$$
$$\frac{d^2\xi}{dt^2} = -CP \sin\theta \,.\, \frac{g}{3a}$$
$$- CP \cos\theta \,.\, \frac{2gl}{3a^2}$$
$$= -\frac{gx}{3a} + \frac{2gl}{3a^2}(a - y).$$

The result given in the text is erroneous in the sign of x.

6. Let the centre of force be origin, and let the friction at the point of contact of the sphere be resolved into X, Y parallel to the co-ordinate axes, these forces being estimated as accelerating forces;

$$\therefore \frac{d^2x}{dt^2} = -P\cos\theta + X,$$
$$\frac{d^2y}{dt^2} = -P\sin\theta + Y. \quad (118, 129).$$

If ω_x, ω_y be the angular velocities of the sphere about

axes through its centre parallel to the co-ordinate axes, then since these diameters are principal axes

$$\left.\begin{aligned} k^2 \frac{d\omega_x}{dt} &= Ya \\ k^2 \frac{d\omega_y}{dt} &= -Xa \end{aligned}\right\}$$

a being the radius of the sphere.

By condition of rolling on a fixed plane the point of contact has no velocity (132);

$$\therefore\ 0 = \frac{dx}{dt} - a\omega_y,$$

$$0 = \frac{dy}{dt} + a\omega_x.$$

$$\therefore\ 0 = \frac{d^2x}{dt^2} - a\frac{d\omega_y}{dt},$$

$$= -P\cos\theta + X + \frac{Xa^2}{k^2},$$

$$= -P\cos\theta + \tfrac{7}{2}X.$$

$$\therefore\ X = \tfrac{2}{7}P\cos\theta.$$

$$\text{So}\ \ Y = \tfrac{2}{7}P\sin\theta.$$

$$\therefore\ \left.\begin{aligned} \frac{d^2x}{dt^2} &= -\tfrac{5}{7}P\cos\theta \\ \frac{d^2y}{dt^2} &= -\tfrac{5}{7}P\sin\theta \end{aligned}\right\}.$$

These are the equations of motion of a particle under a central force $\frac{5}{7}P$, and they lead by the usual method to the equation

$$\frac{d^2u}{d\theta^2} + u = \tfrac{5}{7}\cdot\frac{P}{h^2u^2}.$$

7. The point of contact has no velocity by the condition of rolling. Its two component velocities must therefore be equal and opposite, and they are

(1) the velocity of translation,

(2) the angular velocity multiplied by the radius.

8. If m be the mass of one of the particles, u, v, w its component velocities in direction of the co-ordinate axes, then the velocities of the centre of gravity in the same directions are

$$\frac{\Sigma(mu)}{\Sigma(m)}, \quad \frac{\Sigma(mv)}{\Sigma(m)}, \quad \frac{\Sigma(mw)}{\Sigma(m)}.$$

Now if the velocities of the particles are changed in any manner so that the system may lose the momenta $\Sigma(mu)$, $\Sigma(mv)$, $\Sigma(mw)$ in directions of the co-ordinate axes, the centre of gravity is brought to rest, and the impulses required may be distributed among the particles in an infinite variety of ways, and the problem is indeterminate. But if we suppose moreover that the relative motions of the particles are to remain unaltered, then velocities equal and opposite to those of the centre of gravity are to be destroyed in every body of the system. Thus the velocity u of the particle m parallel to x is to become $u - \frac{\Sigma(mu)}{\Sigma(m)}$, which requires the impulse $-m\frac{\Sigma(mu)}{\Sigma(m)}$ to be applied parallel to x to that particle. So also the impulses $m\frac{\Sigma(mv)}{\Sigma(m)}$, $m\frac{\Sigma(mw)}{\Sigma(m)}$ must be applied to it in negative directions parallel to y and z.

9. Refer the system of fragments to three rectangular co-ordinate axes, and let u, v, w be the velocities in these directions of the fragment m at the moment when it is stopped, the fragment being supposed so small that its motion of translation alone need be considered. The impulses which act upon it are therefore mu, mv, mw, and the motion of the

centre of gravity of the whole is influenced as if impulses equal to these were applied to the whole mass there collected.

If then a parallelogram be drawn of which one side represents in magnitude and direction the previous velocity of the centre of gravity, namely,

$$\frac{1}{\Sigma(m)}\sqrt{(\Sigma\, m\, u)^2 + (\Sigma\, m\, v)^2 + (\Sigma\, m\, w)^2},$$

and another side represents the velocity generated by the impulse $\dfrac{m}{\Sigma(m)}\sqrt{u^2 + v^2 + w^2}$, then that diagonal of the parallelogram which passes through the intersection of these sides represents in magnitude and direction the altered velocity of the centre of gravity of the system.

10. Conservation of areas is not affected by the alteration in the state of the system. If then the values of

$$\Sigma\, m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

before and after the sudden union be equated, and ω represent the new angular velocity,

$$\begin{aligned} 2\, r^2\omega &= 2\, h, \\ &= 2\sqrt{\mu \, . \, a\, b}. \\ \therefore\ \omega &= \frac{a\, b\sqrt{\mu}}{r^2}. \end{aligned}$$

11. Conservation of areas is not disturbed by the explosion, or the value of $\Sigma\, m\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$ is not altered, and this is $\Sigma\left(m\sqrt{\dfrac{b^2}{a}} \, . \cos i\right)$ if the ecliptic be made the plane of xy.

12. During this motion conservation of areas holds good. If then MK^2 be the moment of inertia of the cylinder about its axis and mk^2 that of the shell, ω_1, ω_2 their angular

velocities at the given time, and ω that which they at last obtain and preserve,

$$(MK^2 + mk^2)\,\omega = MK^2\omega_1 + mk^2.\,\omega_2.$$

13. Throughout the motion the system has an invariable plane, and since for the particular shell to which A, ω,...... refer

$$\left.\begin{aligned}\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) &= A\omega l\\ \Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= A\omega m\\ \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) &= A\omega n\end{aligned}\right\}$$

at the beginning of the motion; therefore the equation to the invariable plane is

$$0 = x\,\Sigma\,(A\omega l) + y\,\Sigma\,(A\omega m) + z\,\Sigma\,(A\omega n). \quad (91)$$

Now at last the shells destroy one another's relative motions, and the system moves like a solid sphere, and consequently about an axis perpendicular to this invariable plane already assigned.

These two examples are intended to shew the use of conservation of areas in evading the examination of those mutual actions whereby parts of a system affect one another's motion.

14. If the time be dated from the instant when the centre of the sphere is in the axis of x, the co-ordinates of the molecule m at time t will be

$$x = b\cos\Omega t + r\cos(\omega t + a),$$

$$y = b\sin\Omega t + r\sin(\omega t + a).$$

$$\therefore \frac{dx}{dt} = -b\Omega\sin\Omega t - r\omega\sin(\omega t + a),$$

$$\frac{dy}{dt} = b\Omega\,\text{cosin}\,\Omega t + r\omega\cos(\omega t + a).$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = b^2\Omega + r^2\omega$$

$$+ br\omega \cos \Omega t . \cos (\omega t + a)$$

$$+ br\omega \sin \Omega t . \sin (\omega t + a)$$

$$+ br\omega \sin \Omega t . \cos (\omega t + a)$$

$$+ br\Omega \cos \Omega t . \sin (\omega t + a).$$

$$\text{Now}\ \ \Sigma\, mr \cos (\omega t + a) = 0,$$

$$\Sigma\, mr \sin (\omega t + a) = 0;$$

$$\therefore\ \Sigma\, m \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = Mb^2\Omega + \Sigma\,(mr^2)\,\omega$$

$$= Mb^2\Omega + \tfrac{2}{5} Ma^2 . \omega.$$

15. The instantaneous axis is the line through the point of contact perpendicular to the plane of the wheel, and the angular velocity about this is the same as $\frac{v}{a}$ the angular velocity about the centre of gravity, since each is measured by the angle through which a line fixed in the body revolves. The radius of the sphere is represented by a.

Now *vis viva* of the body $= Mv^2 + Mk^2\frac{v^2}{a^2}$,

vis viva of translation $= Mv^2$,

.... rotation about the instantaneous axis

$$= M(k^2 + a^2)\frac{v^2}{a^2};$$

therefore the sum of these latter exceeds the *vis viva* by Mv^2.

When it is said that the *vis viva* of a body is the sum of its *vires vivæ* of translation and rotation, the rotation must be estimated about the centre of gravity. (154) This example is inserted because it is one where this essential limitation is likely to be overlooked.

16. The principle of conservation of areas applies to this case. Hence if Mk^2 be the moment of inertia of the body about its axis of rotation at any time, and ω the corresponding angular velocity, $Mk^2\omega$ is constant,

$$\therefore\ Mk^2\omega^2 \propto \frac{1}{Mk^2} \propto \frac{1}{\text{moment of inertia}},$$

and $Mk^2\omega^2$ is the *vis viva* of the body (153).

17. Since none of the forces acting on the wheel are of a kind to appear in the equation of *vis viva*, (156) the *vis viva* of the wheel is constant and therefore the velocity also.

The same result may be obtained from the equations of motion, to illustrate the remark of (138) that principles such as that of *vis viva* are nothing but compendious methods of obtaining first integrals of the differential equations. For if x be the distance of the wheel's centre at time t from a given vertical line, θ the angle through which it has revolved from a given position, F the friction, M the wheel's mass,

$$M\frac{d^2x}{dt^2} = F,$$

$$Mk^2\frac{d^2\theta}{dt^2} = -Fa,$$

and by condition of rolling

$$0 = \frac{dx}{dt} - a\frac{d\theta}{dt}.$$

$$\therefore 0 = F + \frac{a^2}{k^2}F;$$

$$\therefore\ F = 0,$$

$$\therefore\ \frac{d^2x}{dt^2} = 0,\quad \frac{d^2\theta}{dt^2} = 0;\quad \therefore\ \frac{dx}{dt} \text{ and } \frac{d\theta}{dt} \text{ are constant.}$$

18. Let x be the distance down which the sphere has descended along the plane, θ the angle through which it has revolved, F the friction acting upwards, i the plane's inclination.

$$\therefore\ M\frac{d^2x}{dt^2} = -F + Mg\sin i,$$

$$Mk^2\frac{d^2\theta}{dt^2} = Fa,$$

and by condition of rolling

$$0 = \frac{dx}{dt} - a\frac{d\theta}{dt}.$$

$$\therefore\ 0 = -F + Mg\sin i - \frac{Fa^2}{k^2},$$

$$F = \frac{Mg\sin i}{1+\frac{5}{2}} = \frac{2Mg\sin i}{7};$$

$$\therefore\ M\frac{d^2x}{dt^2} = \tfrac{5}{7}Mg\sin i,$$

or the sphere's centre moves along the plane as if it were a particle under the uniform accelerating force $\frac{5}{7}g\sin i$, and therefore it describes the length l in the time $\sqrt{\dfrac{14l}{5g\sin i}}$. The unit of time will be that supposed in the numerical measurement of g.

19. In the motion here described the interior cylinder has no motion of rotation, because the only forces acting upon it are its weight and the resultant of the normal pressures between it and the exterior shell, and all the forces act through the axis of the cylinder.

Let m = the mass of the solid cylinder, M = that of the shell, and Mk^2 its moment of inertia about its axis. Then if x be the space down which the system has descended along the plane whose inclination is i,

the *vis viva* of the system

$$= m\left(\frac{dx}{dt}\right)^2 + M\left(\frac{dx}{dt}\right)^2 + \frac{Mk^2}{a^2}\left(\frac{dx}{dt}\right)^2 \qquad (154)$$

$= 2g(M+m)x\sin i$, by the principle of *vis viva*. (Supra, p. 64).

$$\therefore \frac{d^2x}{dt^2} = \frac{(M+m)g\sin i}{M+m+M\dfrac{k^2}{a^2}},$$

so that the centre of gravity of the whole moves like a particle under this constant accelerating force; therefore the time of descending down a length l

$$= \sqrt{\frac{M+m+M\dfrac{k^2}{a^2}}{M+m} \cdot \frac{2l}{g\sin i}}.$$

If however the two bodies were rigidly united, so that the inner cylinder had rotation as well as the shell, then if mk^2 be the moment of inertia of the cylinder, the equation of *vis viva* would be

$$\left(M+m+\frac{Mk^2+mk^2}{a^2}\right)\left(\frac{dx}{dt}\right)^2 = 2(M+m)g\sin i,$$

and the time of descent down l would be

$$\sqrt{\frac{M+m+\dfrac{Mk^2+mk^2}{a^2}}{M+m} \cdot \frac{2l}{g\sin i}}.$$

The ratio of this to the former time

$$= \sqrt{\frac{M+m+\dfrac{Mk^2+mk^2}{a^2}}{M+m+m\dfrac{Mk^2}{a^2}}}.$$

20. Since no horizontal force acts upon the rod and since it moves from rest, its centre of gravity descends in a

vertical line. Let $2a$ be the length of the rod, and θ its inclination to the vertical at time t.

$$\text{Now } \textit{vis viva} \text{ of translation} = M\left\{\frac{d(a\cos\theta)}{dt}\right\}^2,$$

$$= Ma^2\sin^2\theta \cdot \left(\frac{d\theta}{dt}\right)^2.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ rotation} = Mk^2\left(\frac{d\theta}{dt}\right)^2.$$

Of the two forces impressed on the rod, one, the reaction of the plane, does not appear in the equation of *vis viva*, and the centre of gravity has descended through the space

$$a\cos 60^0 - a\cos\theta = \frac{a}{2} - a\cos\theta,$$

$$\therefore\ (a^2\sin^2\theta + k^2)\left(\frac{d\theta}{dt}\right)^2 = 2ga\left(\tfrac{1}{2} - \cos\theta\right);$$

therefore when $\theta = \dfrac{\pi}{2}$, or the rod is horizontal,

$$\text{angular velocity} = \sqrt{\frac{ag}{a^2+k^2}}$$

$$= \sqrt{\frac{ag}{(1+\frac{1}{3})a^2}} = \sqrt{\frac{3g}{4a}}.$$

21. Let the sphere be referred to rectangular axes of x and y in the plane in which its centre moves. Let F, G be the friction at the point of contact in these directions, X, Y the resultant of the rest of the impressed forces similarly resolved, ω_x, ω_y the angular velocities of the sphere about diameters parallel to the co-ordinate axes, a the radius of the sphere.

$$\text{Then}\qquad M\frac{d^2x}{dt^2} = X + F,$$

$$M\frac{d^2y}{dt^2} = Y + G; \quad (118, 129).$$

and since all diameters of the sphere are principal axes,

$$Mk^2 \frac{d\omega_x}{dt} = Ga,$$

$$Mk^2 \frac{d\omega_y}{dt} = -Fa.$$

Since the sphere rolls, its point of contact with the plane has no velocity;

$$\left.\begin{aligned} \therefore\ 0 &= \frac{dx}{dt} - a\omega_y \\ 0 &= \frac{dy}{dt} + a\omega_x \end{aligned}\right\};$$

$$\therefore\ \omega_y \frac{dy}{dt} + \omega_x \frac{dx}{dt} = 0.$$

Hence the instantaneous axis, whose projection on the plane xy is $\dfrac{x'}{\omega_x} = \dfrac{y'}{\omega_y}$, is perpendicular to the direction of motion of the sphere's centre.

Again, $$0 = \frac{d^2x}{dt^2} - a\frac{d\omega_y}{dt},$$

$$0 = X + F + \frac{Fa^2}{k^2}.$$

$$\therefore\ F = -\frac{X}{1 + \dfrac{a^2}{k^2}}.$$

$$\text{So } G = -\frac{Y}{1 + \dfrac{a^2}{k^2}},$$

or the components of the friction are opposite to the components of the other impressed forces and proportional to them.

22. Let the axis of x be horizontal and that of y parallel to the plane. Then the reasoning of the last question gives us the equations of motion required by putting

$$X = 0, \quad Y = g \sin i;$$

$$\therefore \frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2y}{dt^2} = \left(g - \frac{g}{1+\frac{5}{2}}\right) \sin i = \left(g - \frac{2g}{7}\right) \sin i = \frac{5g}{7} \sin i.$$

These are the equations of motion for an ordinary projectile with $\frac{5}{7} g \sin i$ standing in place of g. Therefore the path of the centre of gravity is in general a parabola, the possible exception arising when the sphere is projected either directly up or directly down the plane.

23. Let AB (fig. 25) be the axis of DPA, a cycloid similar to the given cycloid, which the sphere's centre will describe. Let P be the place of the centre at time t, and let the ordinate PM at P cut the generating circle in Q. Let $QAB = \theta$, $2a = AB$.

Now $\quad$ arc $AP = 2AQ$,

$$= 2AB \cos\theta,$$

$$= 4a \cos\theta;$$

therefore the velocity of translation of the sphere

$$= 4a \sin\theta \frac{d\theta}{dt}.$$

Let ω be its angular velocity in space. Then the equation of *vis viva* gives

$$16a^2 \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 + k^2\omega^2 = 2g(h - AM),$$

$$= 2g(h - 2a\cos^2\theta).$$

Now the point of contact must have no velocity by con-

dition of rolling. Hence introducing this condition with reference to the direction QA we have

$$4a \sin\theta \frac{d\theta}{dt} = r\omega,$$

r being the sphere's radius;

$$\therefore \left(16a^2 \sin^2\theta + 16a^2 \sin^2\theta \cdot \frac{k^2}{r^2}\right) \left(\frac{d\theta}{dt}\right)^2 = 2g\,(h - 2a\cos^2\theta),$$

or since $k^2 = \frac{2}{5}r^2$,

$$\frac{56}{5} a^2 \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 = g\,(h - 2a\cos^2\theta).$$

Suppose for simplicity that the motion begins from the highest point of the cycloid;

$$\therefore \frac{56}{5} a \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 = 2g \sin^2\theta,$$

$$\text{or } \left(\frac{d\theta}{dt}\right)^2 = \frac{5 \cdot g}{28 \cdot a},$$

from which the motion may be known.

If $\dfrac{5g}{28a} = \alpha^2$, $\quad \theta = \alpha t$;

therefore time of descent from D to $A = \dfrac{\pi}{2\alpha}$.

24. If x, y be the horizontal and vertical co-ordinates of the sphere's centre at time t, R the reaction of the plane,

$$M\frac{d^2x}{dt^2} = 0, \quad \frac{dx}{dt} = c,$$

$$M\frac{d^2y}{dt^2} = R - Mg.$$

Since the velocity of the plane $\propto t^4$, the space through which it rises in time $t \propto t^5$. If then at^5 be the space through which the plane rises in time t,

$$y = a + at^5; \qquad (1)$$

$$\therefore R - Mg = 20Mat^3.$$

The *vis viva* of the sphere $= M\left\{c^2 + 25a^2t^8\right\}$. But the principle would make the *vis viva*

$$= Mc^2 + 2(R - Mg)\,.\,at^5,$$
$$= Mc^2 + 40Ma^2t^8.$$

Equation (1) shews that the geometrical change of y which the principle of virtual velocities contemplates is zero, while the dynamical change in the course of the actual motion is $5at^4\,.\,\delta t$. The inconsistence of these makes the principle of *vis viva* fail.

At time t let P (fig. 26) be the position of the hanging body whose mass is m, Q that of the body on the table whose mass is m', T the tension of the string, $AP = x$, $AQ = y$, a the unstretched length and $a(1 + et)$ the present length of the string.

Then $$m\frac{d^2x}{dt^2} = mg - T,$$

$$m'\frac{d^2y}{dt} = -T;$$

$$\therefore\ m\frac{d^2x}{dt^2} - m'\frac{d^2y}{dt^2} = mg,$$

$$m\frac{dx}{dt} - m'\frac{dy}{dt} = mgt,$$

the motion beginning from rest, and

$$mx - m'y = \tfrac{1}{2}mgt^2 + mx_0 - m'y_0,$$

x_0y_0 being the initial values of x and y.

But $x + y = (x_0 + y_0)(1 + et)$; (1)

$$\therefore\ (m + m')(x - x_0) = m'aet + \tfrac{1}{2}mgt^2,$$

$$\frac{dx}{dt} + \frac{dy}{dt} = (x_0 + y_0)\,.\,e;$$

$$\therefore\ (m + m')\frac{dx}{dt} = mgt + m'ae,$$

$$(m + m')\frac{dy}{dt} = mae - mgt.$$

$$T = \frac{mm'g}{m + m'}.$$

The *vis viva* of the system

$$= m\left(\frac{dx}{dt}\right)^2 + m'\left(\frac{dy}{dt}\right)^2,$$

$$= \frac{1}{(m + m')^2}\left\{m(m'ae + mgt)^2 + m'(mae - mgt)^2\right\}.$$

But the principle would give the *vis viva*

$$= 2m\int\left(g - \frac{T}{m}\right)dx - 2m'\int\frac{T}{m'}dy,$$

$$= 2mg(x - x_0) - 2\int T(dx + dy),$$

$$= 2mg(x - x_0) - 2ae\int_t T,$$

$$= 2mg(x - x_0) - 2ae\frac{mm'gt}{m + m'},$$

$$= \frac{2mg}{m + m'}(m'aet + \tfrac{1}{2}mgt^2) - 2ae\frac{mm'gt}{m + m'},$$

$$= \frac{m^2g^2t^2}{m + m'}.$$

In this instance the geometrical variation of (1) gives $\delta x + \delta y = 0$, but the dynamical variation gives

$$dx + dy = ae \,.\, \delta t.$$

In the third instance if M be the mass of the expanding sphere, and l the space down which it has fallen, the principle would give the *vis viva* $= 2Mgl$.

Every particle has a velocity $\sqrt{2gl}$ vertically from the motion which it has in common with the centre, and also a velocity v in a direction from the centre arising from the expansion which is taking place. If then the centre be made origin and the axis of z be vertical, m being the mass of a particle at distance r from the centre, the *vis viva* of this particle

$$= m\left\{\frac{v^2x^2}{r^2} + \frac{v^2y^2}{r^2} + \left(\frac{vz}{r} - \sqrt{2gl}\right)^2\right\}$$

$$= m\left(v^2 + 2gl - 2v\sqrt{2gl}\,.\,\frac{z}{r}\right).$$

therefore the *vis viva* of the whole sphere

$$= M(v^2 + 2gl)\,; \qquad \because\ \Sigma\left(\frac{mz}{r}\right) = 0.$$

The distance of two points of the sphere in this case would be given by an expression involving the time t, and thus the geometrical and dynamical changes of this equation would disagree.

In the last of the instances given, as in the case just examined, the equation of *vis viva* would not take into account the velocities of particles of the sphere in consequence of its contraction, and these are components in producing the whole *vis viva* of the body.

25. Let M be the mass of the carriage, m that of each of its four wheels, and mk^2 the moment of inertia of each about its axle, supposing all the four alike, a the radius of each.

If l be the length of plane down which the carriage descends, i its inclination, v the velocity acquired,

$$\left(M + 4m + 4m\,\frac{k^2}{a^2}\right)v^2 = vis\ viva = 2\,(M + 4m)\,gl\sin i.$$

Now let v_1 be the velocity where two wheels on opposite sides cannot turn. Let R be the pressure of each on the plane; therefore μR is the friction on each, and

$$\left(M + 4m + 2m\,\frac{k^2}{a^2}\right)v_1{}^2 = 2\,(M + 4m)\,gl\sin i - 2\mu R\,.\,l.$$

If the plane be smooth and v_2 the velocity then acquired,

$$(M + 4m)\, v_2^2 = 2\,(M + 4m)\, gl \sin i,$$

$$\text{or}\quad v_2^2 = 2gl \sin i.$$

26. Let P descend from the wheel whose radius is r and W ascend by the wheel whose radius is R.

Let P descend through a space x; therefore the wheel belonging to it revolves through the angle $\frac{x}{r}$; therefore the other wheel revolves through the angle $\frac{x}{R}$, and W rises through the space x.

Hence if MK^2 and mk^2 be the moments of inertia of the wheels,

$$\left(P + W + \frac{MK^2}{R^2} + \frac{mk^2}{r^2}\right)\left(\frac{dx}{dt}\right)^2 = 2\,(P - W)\, gx,$$

which determines the motion and proves it to be uniformly accelerated.

27. The efficiency expended in raising any molecule m through the height h is mgh, and therefore the efficiency expended in the operation in question

$$= \text{weight of water} \times h = \sigma Vh,$$

$$\text{therefore efficiency applied} = \frac{\sigma Vh}{a};$$

$$\text{therefore efficiency applied in each minute} = \frac{\sigma Vh}{at},$$

$$= \frac{\sigma Vh}{33000\, at}\ \text{horse powers.}$$

28. Let a be the given exterior radius,

x the interior radius if the body be hollow.

$$\text{Then}\ \frac{\text{moment of inertia about a diameter}}{\text{mass}} = \frac{2}{5} \cdot \frac{a^5 - x^5}{a^3 - x^3}.$$

Now if the globe has rolled down a space s,

$$M\left(1 + \frac{2}{5a^2} \cdot \frac{a^5 - x^5}{a^3 - x^3}\right)\left(\frac{ds}{dt}\right)^2 = vis\ viva$$

$$= 2\,Mgs \sin i\,;$$

therefore the centre of gravity moves like a particle under the constant accelerating force

$$\frac{g \sin i}{1 + \dfrac{2}{5a^2} \cdot \dfrac{a^5 - x^5}{a^3 - x^3}};$$

$$\therefore\ d = \frac{g \sin i}{1 + \dfrac{2}{5a^2} \cdot \dfrac{a^5 - x^5}{a^3 - x^3}} \cdot \frac{n^2}{2}.$$

If now the globe be solid, $d = \dfrac{5gn^2 \sin i}{14}$. If observation does not confirm this equation, the globe must be hollow and x is to be found from the preceding equation.

29. If C (fig. 27) be the centre of the section of the cylinder on which the rods move, every point of the rods describes a circle about C, and the motion is equivalent to that of a body rigidly united to C and oscillating about it. If then G be the centre of gravity of the system, Mk^2 its moment of inertia about an axis through G perpendicular to the plane of motion,

$$\text{time of small oscillation} = \pi \sqrt{\frac{k^2 + CG^2}{g \cdot CG}}.$$

30. In this case the same reasoning shews the same formula to apply.

$$\text{Here}\ \ CG^2 = a^2 - \frac{a^2}{4} = \tfrac{3}{4}a^2,$$

$$k^2 = \frac{a^2}{12};$$

therefore length of simple pendulum $= \dfrac{k^2 + CG^2}{CG}$

$$= \frac{\dfrac{a^2}{12} + \frac{3}{4}a^2}{\dfrac{a}{2}\sqrt{3}} = \frac{\frac{10}{12}}{\frac{1}{2}\sqrt{3}}\, a,$$

$$= \frac{10}{6\sqrt{3}}\, a = \frac{5\sqrt{3}}{9}\, a.$$

31. In the former of the two cases if each string has been turned through an angle θ, since every particle describes a circle, the velocity of every particle is $b\dfrac{d\theta}{dt}$;

$$\therefore\ Mb^2\left(\frac{d\theta}{dt}\right)^2 = \textit{vis viva} = -2Mg\,(h - b\cos\theta);$$

$$\therefore\ b\,\frac{d^2\theta}{dt^2} = -g\sin\theta;$$

therefore the time of small oscillation is $\pi\sqrt{\dfrac{b}{g}}$.

In the latter case if when the rod is displaced horizontally through an angle θ the strings assume an inclination ϕ to the vertical,

the depth of the centre of gravity below the points of support

$$= b\cos\phi.$$

Now $b\sin\phi$ is the horizontal displacement of either end of the rod and $= 2a\sin\dfrac{\theta}{2}$;

$$\therefore\ b\cos\phi\,\frac{d\phi}{dt} = a\cos\frac{\theta}{2}\cdot\frac{d\theta}{dt}.$$

The equation of *vis viva* gives

$$b^2\sin^2\phi\left(\frac{d\phi}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = 2g\,(b\cos\phi - h).$$

If this equation be differentiated and the squares of

$$\phi,\ \theta,\ \frac{d\phi}{dt} \text{ and } \frac{d\theta}{dt} \text{ be neglected,}$$

$$k^2 \frac{d^2\theta}{dt^2}\frac{d\theta}{dt} = -bg\sin\phi\frac{d\phi}{dt}$$

$$= -2ag\sin\frac{\theta}{2}\cdot\frac{a\cos\frac{\theta}{2}}{b\cos\phi}\frac{d\theta}{dt}$$

$$= -\frac{a^2g}{b}\cdot\frac{\sin\theta}{\cos\phi}\frac{d\theta}{dt};$$

$$\therefore\ k^2\frac{d^2\theta}{dt^2} = -\frac{ga^2}{b}\theta, \text{ approximately,}$$

and time of small oscillation

$$= \pi\sqrt{\frac{k^2b}{ga^2}} = \pi\sqrt{\frac{b}{3g}}.$$

32. Let C (fig. 28) be the centre of the surface, and P the ball at time t, O the point of the plane which the sphere originally touched, and Q the point which it now touches;

$$\therefore\ OQ = \tfrac{1}{2}gt^2.$$

Let $CQ = a$, $QCP = \theta$, R the pressure in PC, x, y coordinates of P, m the mass of the ball.

$$\text{Then } m\frac{d^2x}{dt^2} = R\sin\theta,$$

$$m\frac{d^2y}{dt^2} = R\cos\theta - mg.$$

By condition of the ball being in contact with the surface,

$$x = \tfrac{1}{2}gt^2 - a\sin\theta, \quad y = a - a\cos\theta.$$

$$\frac{dx}{dt} = gt - a\cos\theta\frac{d\theta}{dt},$$

$$\frac{d^2x}{dt^2} = g - a\cos\theta\frac{d^2\theta}{dt^2} + a\sin\theta\left(\frac{d\theta}{dt}\right)^2,$$

$$\frac{dy}{dt} = a \sin\theta \frac{d\theta}{dt},$$

$$\frac{d^2y}{dt^2} = a \sin\theta \frac{d^2\theta}{dt^2} + a\cos\theta\left(\frac{d\theta}{dt}\right)^2;$$

$$\therefore\ g\sin\theta = \frac{d^2x}{dt^2}\cos\theta - \frac{d^2y}{dt^2}\sin\theta$$

$$= g\cos\theta - a\frac{d^2\theta}{dt^2},$$

$$a\frac{d^2\theta}{dt^2} = g(\cos\theta - \sin\theta),$$

$$\frac{a}{2}\left(\frac{d\theta}{dt}\right)^2 = C + g\sin\theta + g\cos\theta;$$

$$0 = C + g;$$

$$\therefore\ \frac{a}{2}\left(\frac{d\theta}{dt}\right)^2 = g\sin\theta - g(1-\cos\theta)$$

$$= 2g\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right);$$

$$\therefore\ \frac{d\theta}{dt} \text{ will again } = 0 \text{ when } \frac{\theta}{2} = \frac{\pi}{4},$$

$$\text{or } \theta = \frac{\pi}{2},$$

that is, when the ball has risen into the horizontal plane through C. The ball cannot rise above this point since the forces then acting upon it are the reaction which is horizontal and gravity which is downwards.

33. Let G (fig. 29) be the centre of gravity of the figure of revolution, P the point where it touches the plane, PN the normal at P meeting the axis in N, A the vertex of the figure.

I. Let the plane be smooth. Then G has no horizontal motion. Let y be its height above the plane and θ the angle GNP. Then the equation of *vis viva* gives

$$\left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = 2g(h-y).$$

But P has no vertical velocity,

$$\therefore \frac{dy}{dt} = \frac{d\theta}{dt} . GN . \sin\theta;$$

and N is ultimately the centre of curvature at A,

$$\therefore \frac{dy}{dt} \text{ is inappreciable compared with } \frac{d\theta}{dt};$$

$$\therefore k^2 \left(\frac{d\theta}{dt}\right)^2 = 2g(h - y),$$

$$k^2 \frac{d^2\theta}{dt^2} = -g . GN . \sin\theta = -gc\theta \text{ approximately};$$

$$\therefore \text{time of oscillation} = \pi\sqrt{\frac{k^2}{cg}}.$$

II. If the plane be rough, and x, y be the horizontal and vertical co-ordinates of G, the axis of x lying in the plane, then the equation of *vis viva* is

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = 2g(h - y),$$

and now P has neither horizontal nor vertical velocity;

$$\left.\begin{aligned} \therefore \frac{dy}{dt} &= \frac{d\theta}{dt} . Gm \\ \frac{dx}{dt} &= \frac{d\theta}{dt} . Pm \end{aligned}\right\}, \text{(A)}.$$

Gm being perpendicular to PN,

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 . PG^2$$

$$= \left(\frac{d\theta}{dt}\right)^2 . AG^2 \text{ approximately}.$$

$$\therefore (k^2 + AG^2)\left(\frac{d\theta}{dt}\right)^2 = 2g(h - y),$$

$$(k^2 + AG^2)\frac{d^2\theta}{dt^2} = -g \,.\, Gm,$$

$$= -gc\sin\theta, \text{ approximately.}$$

And the time of oscillation

$$= \pi\sqrt{\frac{k^2 + AG^2}{g \,.\, c}}.$$

Suppose the plane to have a given uniform horizontal motion. The solution of the question in the first case needs no alteration, because the motion of the plane produces no force on the oscillating body. But in the second case the latter equation of system (A) is not true, and the rest of the process ceases to be applicable.

34. Let G (fig. 30) be the centre of gravity of the body, which moves in the vertical line GA because no horizontal force is acting on the body (118), C the centre of its base, P the point of contact. Draw GI perpendicular to PC.

Since G moves vertically the instantaneous axis must meet the plane of the paper in the horizontal line through G.

Since P has horizontal motion, the instantaneous axis must meet the plane of the paper in the vertical line PC.

Hence I designates the place of the instantaneous axis. If the parallelogram $GICB$ be completed,

$$BI = GC = \tfrac{3}{8} \text{ radius};$$

therefore the locus of I is a circle about the fixed point B, and the instantaneous axis lies always on the surface of the corresponding circular cylinder.

36. The arc during the whole of its motion touches the two planes at the same points of them, and every point of the arc retains an invariable distance from the intersection of the normals to the planes at these two points of contact. If O be this intersection, G the centre of gravity of the arc, the motion is similar to that of a body revolving about a fixed axis at O.

Therefore length of equivalent pendulum

$$= \frac{(\text{radius})^2}{OG}$$

$$= \frac{\text{radius} \times \text{arc}}{\text{chord}}.$$

36. The motion of the rods is like that of a body rigidly united to the centre of the surface, and revolving about it. The perpendicular from the centre on the rod $= \frac{a}{2}$; therefore length of simple pendulum

$$= \frac{2\frac{3}{3}a^2 + 2\frac{a^2}{4}}{2\frac{a}{2}}$$

$$= 2a.$$

37. Let O (fig. 31) be the centre of the fixed sphere, C the centre of the base and G the centre of gravity of the hemisphere, P its point of contact at time t when CG makes an angle θ with the vertical. Draw PK vertical. Let R, r be the radii of the sphere and hemisphere;

$$\therefore\ CK = R\theta \cot\theta.$$

Now if O be origin, the co-ordinates of G are

$$\left.\begin{aligned} &R\sin\theta - GK\sin\theta \\ &= R\sin\theta - (CK - CG)\sin\theta \\ &= (R + CG)\sin\theta - R\theta\cos\theta \end{aligned}\right\}\ \text{horizontally,}$$

$$\text{and}\ \ (R + CG)\cos\theta + R\theta\sin\theta\ \ \text{vertically;}$$

therefore the *vis viva* of translation is

$$M\left\{(R + CG)\cos\theta + R\theta\sin\theta - R\cos\theta\right\}^2 \left(\frac{d\theta}{dt}\right)^2$$

$$+ M\left\{(R + CG)\sin\theta - R\theta\cos\theta - R\sin\theta\right\}^2 \left(\frac{d\theta}{dt}\right)^2$$

$$= M\left\{CG^2 + R^2\theta^2\right\}\left(\frac{d\theta}{dt}\right)^2.$$

Therefore by the principle of *vis viva*,

$$(k^2 + CG^2 + R^2\theta^2)\left(\frac{d\theta}{dt}\right)^2$$

$$= 2g\left\{h - (R + CG)\cos\theta - R\theta\sin\theta\right\}.$$

By differentiation

$$\left\{k^2 + CG^2 + R^2\theta^2\right\}\frac{d^2\theta}{dt^2} + R^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$= g\left\{(R + CG)\sin\theta - R\sin\theta - R\theta\cos\theta\right\}.$$

Now if the disturbance be such that the initial and all succeeding values of θ^2 may be omitted, since $\left(\frac{d\theta}{dt}\right)^2$ is of this order, we have

$$(k^2 + CG^2)\frac{d^2\theta}{dt^2} = -g(R - CG)\theta,$$

and the time of small oscillation $= \pi\sqrt{\frac{\frac{2}{5}r^2}{g(R - \frac{3}{8}r)}}$,

supposing the hemisphere homogeneous. If $8R$ is less than $3r$, the equilibrium is unstable.

In the second case, with a similar construction and notation, fig. 32,

$$\frac{KPC}{KCP} = \frac{r}{R}; \quad \therefore \frac{KPC}{\theta} = \frac{r}{R + r};$$

$$\therefore CK = r \cdot \frac{\sin\left(\frac{r\theta}{R + r}\right)}{\sin\theta}$$

and the horizontal and vertical co-ordinates of G are

$$\left.\begin{array}{l}(R + r)\sin\left(\frac{r\theta}{R + r}\right) - CG\sin\theta \\ (R + r)\cos\left(\frac{r\theta}{R + r}\right) - CG.\cos\theta\end{array}\right\};$$

therefore by the principle of *vis viva*

$$\left(k^2 + r^2 + CG^2 - 2r \,.\, CG . \cos \frac{R\theta}{R+r}\right) \left(\frac{d\theta}{dt}\right)^2$$

$$= 2g \left\{h - (R+r) \cos \frac{r\theta}{R+r} + CG . \cos\theta\right\}.$$

Whence, as before,

$$\{k^2 + (r - CG)^2\} \frac{d^2\theta}{dt^2} = -g\left(CG - \frac{r^2}{R+r}\right)\theta,$$

and the time of oscillation is $\pi \sqrt{\dfrac{k^2 + (r - CG)^2}{\left(CG - \dfrac{r^2}{R+r}\right) g}}$,

$$\text{or } \pi \sqrt{\frac{26}{5} . \frac{(R+r)\, r}{(3R - 5r)\, g}}.$$

If $3R$ is less than $5r$, the equilibrium is unstable.

38. Let horizontal and vertical axes be drawn from the line about which the plank turns, and at time t let x, y be co-ordinates of the centre of the sphere whose radius is a and mass m, θ the angle through which it has turned in space, ϕ the angle through which the plank has turned, M the mass of the plank, R, F the mutual actions perpendicular to the plank and along it, r the distance of the point of contact from the origin.

Then (1) for the motion of the plank,

$$MK^2 \frac{d^2\phi}{dt^2} = Rr;$$

and (2) for the motion of the ball,

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= R\sin\phi - F\cos\phi, \\ m\frac{d^2y}{dt^2} &= mg - R\cos\phi - F\sin\phi, \\ mk^2\frac{d^2\theta}{dt^2} &= Fa. \end{aligned}\right\}$$

Since the ball rolls, its point of contact has no velocity in direction of the plank (132),

$$\therefore\ 0 = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\phi - a\frac{d\theta}{dt}.$$

Also from the ball being in contact with the plank,

$$\left.\begin{aligned} x &= r\cos\phi + a\sin\phi \\ y &= r\sin\phi - a\cos\phi \end{aligned}\right\}.$$

So that seven equations are obtained for assigning the seven quantities x, y, θ, ϕ, r, F, R whose determination in terms of t defines the motion.

Vis viva gives one integral

$$MK^2\left(\frac{d\phi}{dt}\right)^2 + m\left(\frac{dx}{dt}\right)^2 + m\left(\frac{dy}{dt}\right)^2 + mk^2\left(\frac{d\theta}{dt}\right)^2$$
$$= 2mg(y-h),\ (153,\ 154);$$

but the solution if pursued further becomes tedious.

39. Let the plane be made that of xy, and let the fixed ring A (fig. 33) lie in the axis of z; let $OM = x$, $MN = y$, $NG = z$ be the co-ordinates of the centre of gravity of the rod, $NOx = \theta$, $OAG = \phi$.

Since the rod has two rotations about its centre of gravity, one about GN which the change in the angle θ exhibits, and the other about a line perpendicular to the plane GAO which the change in ϕ exhibits; therefore by the principle of *vis viva*,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + k^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + k^2\left(\frac{d\phi}{dt}\right)^2$$
$$= 2g(h-z),\ (153,\ 154),$$

$k^2\sin^2\phi \times$ mass of rod being the moment of inertia of the rod about GN.

The principle of conservation of areas applies to the line Oz,

$$\therefore\ (k^2 \sin^2 \phi + x^2 + y^2)\,\frac{d\theta}{dt} \propto 1.$$

$$\text{Let }\ x^2 + y^2 = r^2,$$

$$\text{then }\ x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$z = a\cos\phi,$$

$$\text{if }\ 2a = \text{length of the rod.}$$

$$\therefore\ \left(\frac{dr}{dt}\right)^2 + (r^2 + k^2\sin^2\phi)\left(\frac{d\theta}{dt}\right)^2 + (k^2 + a^2\sin^2\phi)\left(\frac{d\phi}{dt}\right)^2$$

$$= 2g\,(h - a\cos\phi),$$

$$\text{and }\ (k^2\sin^2\phi + r^2)\,\frac{d\theta}{dt} \propto 1.$$

If ϕ were eliminated between these, a differential equation in r and θ results which defines the path of the projection of G.

40. Let Ox (fig. 34) be the plane along which the wheel rolls, C its centre and P its point of contact at time t. F the friction, $OP = x$, θ the angle through which the wheel has revolved, M its mass, $CP = a$;

$$\therefore\ M\frac{d^2x}{dt^2} = -F,$$

$$Mk^2\frac{d^2\theta}{dt^2} = Fa,$$

where F is a known quantity $= \mu$. weight, μ being the coefficient of friction, and it will retain this value until sliding ceases (132);

$$\therefore\ M\frac{dx}{dt} = MV - Ft,$$

$$Mk^2\frac{d\theta}{dt} = Mk^2\omega + Fat,$$

if V and ω be the initial velocities of translation and rotation.

Since P moves horizontally, the instantaneous axis meets the plane of the paper in the vertical through P. Let y be its distance from P;

$$\therefore\ \frac{dx}{dt} = (a - y)\frac{d\theta}{dt}.$$

$$\therefore\ V - \frac{F}{M}.t = \frac{a-y}{k^2}\left\{k^2\omega + \frac{F}{M}at\right\}.$$

$$\text{Also } x = Vt - \tfrac{1}{2}\frac{F}{M}.t^2,$$

$$\text{if } x = 0 \text{ when } t = 0,$$

$$\therefore\ \frac{F}{M}t\left\{1 + \frac{a(a-y)}{k^2}\right\} = V - (a-y)\omega.$$

$$\therefore\ x = V.\frac{M}{F}.\frac{V-(a-y)\omega}{1+\dfrac{a^2-ay}{k^2}} - \tfrac{1}{2}\frac{M}{F}.\left\{\frac{V-(a-y)\omega}{1+\dfrac{a^2-ay}{k^2}}\right\}^2.$$

The locus of the instantaneous axis is the cylinder whose directrix is defined by this equation.

41. Let C (fig. 35) be the vertex of the cone, CA its vertical axis, G the centre of gravity of one half of it, R, S the vertical and horizontal actions at C initially upon one half, and ϕ the initial angular accelerating force on the same. The motion of this half of the cone being that of a body turning about a fixed point C,

$$\therefore\ (k^2 + CG^2)\,\phi = g.GH.\quad (41)$$

Now if x be the horizontal co-ordinate of G when CG makes an angle θ with the vertical,

$$x = CG.\sin\theta,$$

$$\frac{dx}{dt} = CG.\cos\theta\frac{d\theta}{dt},$$

$$\frac{d^2x}{dt^2} = CG.\cos\theta\frac{d^2\theta}{dt^2} - CG.\sin\theta\left(\frac{d\theta}{dt}\right)^2,$$

$$= \phi.CH \text{ initially}.$$

$$\text{So } \frac{d^2y}{dt^2} = -\phi . GH.$$

$$\left.\begin{aligned} \therefore\ M.CH.\phi &= S, \\ -M.GH.\phi &= R - Mg. \end{aligned}\right\}$$

$2M$ being the mass of the whole cone.

$$\text{Now } k^2 + CG^2 = \tfrac{3}{4} CA^2, \quad CH = \tfrac{3}{4} CA, \quad GH = \frac{CA}{\pi}.$$

$$\therefore\ \phi = \frac{4g}{3\pi . CA}.$$

$$S = \frac{Mg}{\pi} = \frac{\text{weight of whole cone}}{2\pi}.$$

$$\frac{Mg - R}{Mg} = \frac{4}{3\pi^2}.$$

42. Let V be the velocity of the impinging ball which is changed into V' by the collision, m the mass of each ball, u, v the initial horizontal and vertical velocities of either of the three balls on the table. Let R' be the reaction of the table on each of these latter, R the action between each of them and the falling ball in a direction making an angle θ with the vertical.

Then for the motion of the upper ball,

$$m(V - V') = 3R\cos\theta;$$

and for the motion of each lower ball,

$$\left.\begin{aligned} mu &= R\sin\theta, \\ mv &= R' - R\cos\theta. \end{aligned}\right\}$$

When the compression is completed so that two bodies in contact have the same velocity in direction of their common normal (134), we have,

$$v = 0,$$

$$V'\cos\theta = -v\cos\theta + u\sin\theta.$$

If then R_1 be the impulsive action during the compression,

$$m V \cos\theta = 3R_1 \cos^2\theta + R_1 \sin^2\theta,$$

$$R_1 = \frac{m V \cos\theta}{1 + 2\cos^2\theta};$$

therefore if e be the elasticity,

$$R = (1 + e) R_1;$$

$$\therefore\ (V - V') = 3(1 + e)\frac{V \cos^2\theta}{1 + 2\cos^2\theta}.$$

Now the centres of the spheres form a regular tetrahedron, and $\cos^2\theta = \frac{2}{3}$;

$$\therefore \text{ if } V' = 0,$$

$$1 + \tfrac{4}{3} = 3(1 + e) \cdot \tfrac{2}{3},$$

$$7 = 6(1 + e),$$

$$e = \tfrac{1}{6}.$$

If T be the impulsive tension of the string by which the horizontal velocity of each of the lower balls is at once destroyed, the equations of motion would be

$$m(V - V') = 3R\cos\theta,$$

$$0 = R\sin\theta - 2T\cos 30^\circ,$$

$$mv = R' - R\cos\theta.$$

If the balls be inelastic, $v = 0$, $V' = 0$,

$$R_1 = \frac{mV}{3\cos\theta}.$$

therefore if e be the elasticity

$$T\sqrt{3} = R\sin\theta,$$

$$= (1 + e) R_1 \sin\theta,$$

$$= (1 + e)\frac{mV\tan\theta}{3},$$

$$= \frac{1 + e}{6} \cdot mV\sqrt{2}.$$

$$\therefore\ T = \frac{(1 + e)\sqrt{2}}{6\sqrt{3}} mV.$$

43. The statement of this problem when corrected will be thus:

Two equal smooth balls moving in parallel straight lines with a common velocity V perpendicular to the line joining their centres impinge at once on a third equal ball lying at rest, so that the centres of the three at the moment of impact form an equilateral triangle in the plane of the motion of the two balls. The quiescent ball will begin to move with the velocity $\frac{3}{5}(1+e)V$.

Let m be the mass of each ball, and let R be the blow between each of the moving balls and the quiescent ball, generating in the latter a velocity V'. Let v be the velocity of each of the other balls after impact in the direction of their previous motion, u the velocity in a direction perpendicular to this;

$$\therefore\ m(V-v) = R\cos 30^0,$$

$$mu = R\sin 30^0,$$

$$mV' = 2R\cos 30^0.$$

If the balls were inelastic, then when the compression was complete we should have

$$V'\cos 30^0 = v\cos 30^0 - u\sin 30^0;$$

therefore if R_1 be the impulse in this case,

$$2R_1\cos^2 30 = mV\cos 30^0 - R_1\cos^2 30^0 - R_1\sin^2 30^0,$$

$$R_1 = \frac{mV\cos 30^0}{1+2\cos^2 30} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{3}{2}}mV = \frac{\sqrt{3}}{5}mV.$$

therefore if e be the elasticity, $R = (1+e)R_1$,

$$\text{and } V' = \frac{R\sqrt{3}}{m} = \frac{3}{5}(1+e)V.$$

44. Let A, B (fig. 36) be the centres of the balls, m the mass of each, AC the direction in which A is required to move, and AP the direction in which for this purpose it must be struck by the blow P, R the mutual impulse between A and B.

Let $CAB = a$, and $v \cos a$, and $v \sin a$, the component velocities produced in A in and perpendicular to AB, u the velocity generated in B, $CAB = \theta$;

$$\left.\begin{aligned} \therefore\ mu &= R, \\ mv \cos a &= P \cos \theta - R, \\ mV \sin a &= P \sin \theta. \end{aligned}\right\}$$

When the compression is complete

$$u = v \cos a\,;$$

therefore if R_1 be then the value of R,

$$R_1 = P \cos \theta - R_1,$$

$$R_1 = \tfrac{1}{2} P \cos \theta\,;$$

therefore if e be the elasticity, $R = \dfrac{1+e}{2} \,.\, P \cos \theta$;

$$\therefore\ \tan a = \frac{P \sin \theta}{P \cos \theta - \frac{1}{2}(1+e) \,.\, P \cos \theta},$$

$$= \frac{\tan \theta}{\frac{1}{2} - \frac{1}{2} e} = \frac{2 \tan \theta}{1 - e}.$$

45. Let C (fig. 37) be the centre of the sphere which touches the horizontal plane in B, A the fixed point. Let a be the radius of the sphere, ω its angular velocity, and $a\omega$ its linear velocity, which are suddenly destroyed; R, F the normal and tangential impulses at A, and R', F' those at B, $ACB = \dfrac{\pi}{2} + a$;

$$\therefore\ Ma\omega = F' + R\cos a + F\sin a,$$

$$0 = R' + F\cos a - R\sin a,$$

$$Mk^2\omega = (F - F')\,a.$$

Hence unless the relation between the normal and tangential actions at one at least of the points A, B is assigned, the problem is indeterminate, and the motion may be destroyed under different distributions of the normal and tangential action at the two points.

46. Let AB (fig. 38) be the rod whose centre C is struck. Let U, V be its velocities of translation in and perpendicular to AB, Ω its angular velocity, and let V become V'. Let u, v, ω be similar motions for the rod CE, and let these become u', v', ω'; let M, m be the masses of AB, CE. Let R be the blow at C perpendicular to AB, $DCB = a$;

$$\left.\begin{aligned} \therefore\ M(V' - V) &= R, \\ m(u' - u) &= 0, \\ m(v' - v) &= -R, \\ mk^2(\omega' - \omega) &= -R\,.\,CD\cos a. \end{aligned}\right\}\ (1)$$

If the rods were inelastic, we should have

$$V' = v' + \omega'\,.\,CD\cos a,$$

$$V + \frac{R_1}{M} = v - \frac{R_1}{m} + \frac{CD\cos a}{mk^2}(mk^2\omega - R_1\,.\,CD\cos a),$$

$$R_1\left(\frac{1}{M} + \frac{1}{m} + \frac{CD^2}{mk^2}\right) = v - V + CD\,.\cos a\,.\,\omega.$$

Then $R = (1 + e)\,R_1$ and every element of the motion after impact becomes known by means of equations (1).

47. Let V be the velocity of translation of the cylinder which becomes V' by the horizontal impulse F, while an

angular velocity $\frac{V'}{a}$ is also generated. [See Prob. 7. Page (68)];

$$\therefore \; M(V - V') = F,$$

$$Mk^2 \frac{V'}{a} = Fa\,;$$

$$\therefore \; \frac{V - V'}{V'} = \frac{k^2}{a^2} = \tfrac{1}{2},$$

$$\frac{V}{V'} = 1 + \tfrac{1}{2} = \tfrac{3}{2},$$

$$V' = \tfrac{2}{3} V.$$

48. Let X, Y be the horizontal and vertical components of the blow applied at a point where the radius makes an angle α with the horizon, R the normal action, F the tangential action where the wheel meets the plane, a the radius, v, $\frac{v}{a}$ the velocities of translation and rotation before impact, which become v' and $\frac{v'}{a}$ afterwards;

$$\therefore \; M(v' - v) = X - F,$$

$$0 = R - Y,$$

$$\frac{Mk^2}{a}(v' - v) = Fa + Xa \sin\alpha - Ya\cos\alpha.$$

$$\text{Now } k^2 = a^2,$$

$$\therefore \; X - F = F + X\sin\alpha - Y\cos\alpha,$$

$$2F = X(1 - \sin\alpha) + Y\cos\alpha,$$

the condition required, in addition to that of Y being equal to R and therefore necessarily positive.

49. Let $2a$ be the length of the rod, ω the angular velocity before impact about the centre of gravity (fig. 39); therefore $a\omega \sin\alpha$, $a\omega\cos\alpha$ are the horizontal and vertical velocities of the centre of gravity.

These velocities are all destroyed by these three impulses, R' horizontally at B, R the vertical and T the horizontal impulse at A;

$$\therefore\ Ma\omega \sin \alpha = T - R',$$

$$Ma\omega \cos \alpha = R,$$

$$M\frac{a^2}{3}\omega \quad = Ta \sin \alpha + R'a \sin \alpha - Ra \cos \alpha;$$

$$\therefore\ \tfrac{1}{3} Ma^2\omega = 2R'a \sin \alpha + Ma^2\omega \sin^2 \alpha - Ma^2\omega \cos^2 \alpha.$$

Now R' cannot be negative;

$$\therefore\ \tfrac{1}{3} + \cos^2 \alpha - \sin^2 \alpha \text{ cannot be negative};$$

$$\tfrac{1}{3}(1 + \tan^2 \alpha) + 1 - \tan^2 \alpha > 0,$$

$$4 - 2\tan^2 \alpha > 0,$$

$$\therefore\ \tan^2 \alpha < 2.$$

50. Let V be the vertical velocity of G the centre of gravity of AB before it impinges on the peg C, (fig. 40), ω the angular velocity about G, and let v, u be the vertical and horizontal velocities of G after impact, ω' the angular velocity of the rod, R the impulse at C perpendicular to the rod, R' the vertical impulse at A, α the rod's inclination to the horizon, M the mass of the rod;

$$\left.\begin{aligned} \therefore\ M(V - v) &= R' + R \cos \alpha \\ Mu \quad &= R \sin \alpha \\ Mk^2(\omega - \omega') &= R \,.\, GC - R' \,.\, AG \cos \alpha \end{aligned}\right\}.$$

If the rod be inelastic, then after the collision A has no vertical velocity and C has no velocity perpendicular AB;

$$\therefore\ v = \omega' \,.\, GA \,.\, \cos \alpha,$$

$$v \cos \alpha - u \sin \alpha = \omega' \,.\, GC;$$

$$\therefore\ MV - R' - R\cos\alpha = GA \,.\cos\alpha \left(M\omega - \frac{R \,.\, GC - R' \,.\, AG \cos \alpha}{k^2}\right),$$

$$(MV - R' - R\cos\alpha)\cos\alpha - R\sin^2\alpha = CG\left(M\omega - \frac{R \,.\, GC - R' \,.\, AG \cos\alpha}{k^2}\right).$$

These equations give R, R' and then the elements of the motion immediately after impact are known.

51. Let the rod AB (fig. 41) revolving upon C impinge with an angular velocity ω upon the peg D, and have immediately afterwards an angular velocity ω' in consequence of the impulses R at D, R' at C.

$$M.CG(\omega+\omega')=R+R',$$
$$Mk^2(\omega-\omega')=(R-R')\,CG.$$

The problem now assumes two forms according as the motion is wholly destroyed or the rod turns about D.

I. If motion ceases, or $\omega'=0$,

$$M(CG^2-k^2)\,\omega=2R'.CG.$$

Since R' cannot be negative

$$CG^2>k^2>\frac{AG^2}{3},\quad CG>\frac{AG}{\sqrt{3}}.$$

II. If motion continues and the rod rises from C, $R'=0$,

$$\therefore\ 2M\omega'=\frac{R}{k^2.CG}(k^2-CG^2),$$
$$\therefore\ CG<\frac{AG}{\sqrt{3}}.$$

The cases are therefore distinguished by CD being $>$ or $<\dfrac{AB}{\sqrt{3}}$.

52. Let u be the velocity of translation after impact in direction of the plane and upwards, therefore $\dfrac{u}{a}$ the angular velocity after impact, F the impulse along the plane.

$$\therefore\ M(u+V\cos i)=F,$$
$$Mk^2\left(\omega-\frac{u}{a}\right)=Fa.$$

$$\therefore \frac{k^2}{a^2}(a\omega - u) = u + V\cos i,$$

$$a\omega - u = 2u + 2V\cos i,$$

$$3u = a\omega - 2V\cos i.$$

therefore the disc runs $\begin{cases}\text{upwards}\\ \text{downwards}\end{cases}$

as u is $\begin{cases}\text{positive}\\ \text{negative},\end{cases}$

as $\frac{1}{2}aw \gtrless V\cos i$.

53. The difficulty of this problem consists in duly recognizing the impulsive action which is constantly going on whereby new portions of the chain are being suddenly endued with finite velocity.

Now let us suppose in place of the continuous chain that small portions of its mass, each $= m\delta$, are collected upon a weightless thread at intervals δ, and at time t let v be the velocity of the system, P the mass of the suspended weight, x the length of that portion of string which with its attached weights has been already drawn up, and let another elementary weight be on the point of being put in motion. Now in putting this new element into motion an impulse arises, and the velocity of the system will be diminished by δv suppose, so that its momentum is unaltered,

$$\therefore (mx + P)v = (mx + m\delta + P)(v - \delta v),$$

or neglecting squares of small quantities,

$$\delta v = \frac{mv\delta}{P + mx}.$$

The accelerating force is now $\dfrac{P - mx - m\delta}{P + mx + m\delta}.g$, and under this the system moves through the space δ and has its velocity changed from $v - \delta v$ to $v + dv$ suppose;

$$\therefore\ (v + dv)^2 = (v - \delta v)^2 + 2\,\frac{P - mx - m\delta}{P + mx + m\delta}\cdot g\delta,$$

or neglecting squares as before we have,

$$2vdv = -\frac{2mv^2}{P + mx}\,\delta + 2\,\frac{P - mx}{P + mx}.g.\delta.$$

Here dv is the entire increment of velocity while the system moves through the space δ. If now we bring our hypothesis to the limit, when the hypothetical string and attached weights on which we have reasoned becomes the continuous chain of the question before us, then, putting $\frac{dv}{dx}$ as the limit of $\frac{dv}{\delta}$ we have

$$d(v^2) + \frac{2m}{P + mx}v^2 = 2\,\frac{P - mx}{P + mx}g.$$

Multiply by $(P + mx)^2$ and

$$\frac{d(P + mx)^2 v^2}{dx} = 2(P^2 - m^2x^2)g,$$

$$\therefore\ (P + mx)^2 v^2 = C + 2P^2gx - \tfrac{2}{3}m^2gx^3.$$

$$v^2 = \tfrac{2}{3}\cdot\frac{3P^2gx - m^2gx^3}{(P + mx)^2},$$

if $x = 0$ when motion begins.

If the chain lies originally in a horizontal line along the table, the whole of it moves together, and no impulsive action takes place. Let the projection of the pulley on the plane be the origin of horizontal and vertical co-ordinates of x and y, T the uniform tension of the string by which P hangs and z the depth of P below the pulley at time t, T' the tension of the chain at the point x, y, to which point s is the distance from the pulley measured along the curve, T_1 the tension of the chain where it meets the table, and η the distance between the vertical line through the pulley and the more distant end of the chain on the table, l the length of the chain.

The motion of P gives the equation

$$P \frac{d^2 z}{dt^2} = Pg - T,$$

and from considering an element of the chain of which k is the mass of an unit of length, we have

$$\left. \begin{aligned} k \frac{d^2 x}{dt^2} &= \frac{d\left(T' \dfrac{dx}{ds}\right)}{ds} \\ k \frac{d^2 y}{dt^2} &= -g - \frac{d\left(T' \dfrac{dy}{ds}\right)}{ds} \end{aligned} \right\}$$

while from the motion of the part in contact with the table we have

$$k(\eta - x_1) \frac{d^2 \eta}{dt^2} = -T_1.$$

It does not appear possible to carry the solution further than this statement of the equations on which, together with geometrical conditions, it depends. If they could be solved, we should have to introduce the conditions that T' becomes T where the chain is attached to the string and becomes T_1 where the chain meets the table at the distance x_1 from the origin

54. Let E (fig. 42) be the point where the rod presses upon the plane, A, B the rings which constrain it, DF the plane. Let the rod's direction meet the horizontal plane in C. Let M, m be the masses of the wedge and rod, $CE = y$, $CD = x$, α the wedge's inclination to the horizon.

The principle of *vis viva* gives

$$M\left(\frac{dx}{dt}\right)^2 + m\left(\frac{dy}{dt}\right)^2 = 2mg(h - y).$$

$$\text{But } y = x \tan \alpha;$$

$$\therefore (M + m \tan^2 \alpha)\left(\frac{dx}{dt}\right)^2 = 2mg(h - y).$$

Therefore when $y = 0$,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2mgh}{M + m\tan^2 a}.$$

The square root of this is the velocity required.

$$\text{Also}\quad (m + M\cot^2 a)\left(\frac{dy}{dt}\right)^2 = 2mg(h - y);$$

$$\therefore\ \frac{d^2y}{dt^2} = -\frac{mg}{m + M\cot^2 a},$$

the effective force on every particle of the rod.

Let then R be the action at E, S, S' the actions of the rings in the directions which the figure exhibits;

$$\therefore\ S = S' + R\sin a,$$

$$m\,.\,\frac{mg}{m + M\cot^2 a} = mg - R\cos a,$$

$$S\,.\,EB = S'\,.\,EA. \qquad (1)$$

$$R\cos a = mg\left(1 - \frac{m}{m + M\cot^2 a}\right)$$

$$= \frac{Mmg\cot^2 a}{m + M\cot^2 a};$$

$$\therefore\ S - S' = \frac{Mmg\cot a}{m + M\cot^2 a}. \qquad (2)$$

Equations (1), (2) give S and S'.

Secondly, let the rod be descending with a given velocity v when the system is suddenly brought to rest. Let R, S, S' now represent impulsive forces in the directions in which those letters have denoted the finite forces in the previous part of the problem;

$$\therefore\ mv = R\cos a,$$

$$S = S' + R\sin a,$$

$$S\,.\,EB = S'\,.\,EA.$$

When R is eliminated, S and S' are determined.

55. Let D, E (fig. 43) be the pegs, O the middle point between them, θ the inclination of either to the vertical CO at time t, $2a$ the length of each, R the reaction of D or E, X the reaction at C which is horizontal.

$$\therefore\ 2M\frac{d^2(DO\cot\theta - a\cos\theta)}{dt^2} = 2R\sin\theta - 2Mg,$$

$$M\frac{d^2(DO - a\sin\theta)}{dt^2} = R\cos\theta + X,$$

$$Mk^2\frac{d^2\theta}{dt^2} = R(DO - a\sin\theta)\operatorname{cosec}\theta - Xa\cos\theta.$$

These are the equations of motion, and by the principle of *vis viva* an integral of them gives

$$\textit{vis viva} \text{ of the system} = 4Mg(h - DO\cot\theta + a\cos\theta).$$

Therefore when the greatest opening is attained, θ is given by the equation

$$h + a\cos\theta = DO\,.\cot\theta.$$

If T be the impulsive tension when the string is tight, this force together with R, R, X the impulsive actions at D, E, C, brings the system to rest, and therefore destroys the angular velocity (ω), and the linear velocities of the centre of gravity of each rod, which are

$$\begin{cases}(DO\operatorname{cosec}^2\theta_1 - a\sin\theta_1)\,\omega \text{ downwards,}\\ \qquad a\cos\theta_1\,.\,\omega \text{ horizontally,}\end{cases}$$

θ_1 being the value of θ at the time of this event.

$$\therefore\ M(DO\operatorname{cosec}^2\theta_1 - a\sin\theta_1)\,\omega = R\sin\theta_1,$$

$$Ma\cos\theta_1\,.\,\omega = X + R\cos\theta_1 - T,$$

$$Mk^2\omega = (T + X)\,a\cos\theta_1 - R(DO - a\sin\theta_1)\operatorname{cosec}\theta_1.$$

These equations assign the impulses X, R, T.

56. Let M be the mass, $2a$ the length of each of the rods, ϕ the inclination of each to the vertical at time t, m the mass of the sphere, r its radius, and therefore $\dfrac{r}{\sin\phi}$ the height of its centre above the joint.

The principle of *vis viva* gives

$$m\left\{\frac{d(r\operatorname{cosec}\phi)}{dt}\right\}^2 + 3M(a^2+k^2)\left(\frac{d\phi}{dt}\right)^2$$

$$= 2mg(h - r\operatorname{cosec}\phi) + 6Mg(h' - a\cos\phi),$$

or if $\phi = \alpha$ initially,

$$\left\{m\operatorname{cosec}^2\phi\cot^2\phi + 3M(a^2+k^2)\right\}\left(\frac{d\phi}{dt}\right)^2$$

$$= 2mgr(\operatorname{cosec}\alpha - \operatorname{cosec}\phi) + 6Mga(\cos\alpha - \cos\phi).$$

Now let ϕ have the value ϕ_1 when the motion is destroyed. Let ω be the angular velocity of the rods and v the velocity of the sphere just before the impulse, T the impulsive tension of each string, R the impulse on each rod;

$$\left.\begin{aligned}\therefore\ 3R\sin\phi_1 &= mv,\\ M(k^2+a^2)\,\omega + Rr\cot\phi_1 &= 2T\cos 30^0.2a\cos\phi_1,\end{aligned}\right\}$$

whence R being eliminated, T is assigned.

57. Let the normal at P (fig. 44) where the impact takes place meet the axis a in G at the angle α, and let PM be perpendicular to a. Let v', u' be the velocities of O after impact in directions of a and b, R the impulse;

$$\therefore\ Mk^2\omega = R.OG.\sin\alpha,$$

$$M(v - v') = R\cos\alpha,$$

$$Mu' = -R\sin\alpha,$$

and if V be the velocity in direction GP generated in the sphere,

$$MV = R.$$

Since there is no elasticity

$$V = v' \cos a + u' \sin a,$$

$$R = MV \cos a - R \cos^2 a - R \sin^2 a,$$

$$R = \tfrac{1}{2} Mv \cos a\,;$$

$$\therefore\ \tfrac{1}{2} Mv . OG \sin a \cos a = Mk^2\omega.$$

Now $OG = ae^3$, $\tan a = \dfrac{1}{e}$,

$$k^2 = \frac{a^2 + b^2}{5},$$

$$\therefore\ \frac{2a\omega}{v} = \frac{5e^4}{(2 - e^2)(1 + e^2)} = 1, \text{ if } e^2 = \tfrac{2}{3}.$$

58. Let z be the height of the ring above the lowest point of the hoop at time t, and r its distance from the axis of revolution, ω the angular velocity, ϕ the angle which r subtends at the centre of the hoop.

Now the effective forces in the plane of the hoop are

$$\left.\begin{array}{l} \dfrac{d^2z}{dt^2} \text{ upwards} \\[2ex] \text{and } \dfrac{d^2r}{dt^2} - r\omega^2 \text{ in direction of } r \end{array}\right\}.$$

These reversed must balance the other forces in the plane of the hoop, viz.: gravity and the part of the hoop's reaction in that plane;

$$\therefore\ \left(\frac{d^2r}{dt^2} - r\omega^2\right)\cos\phi + \frac{d^2z}{dt^2}\sin\phi + g\sin\phi = 0,$$

while $r = a\sin\phi$, $z = a - a\cos\phi$.

$$\therefore\ a\frac{d^2\phi}{dt^2} + g\sin\phi - a\omega^2\sin\phi\cos\phi = 0\,;$$

$$\therefore\ a\left(\frac{d\phi}{dt}\right)^2 = (\cos\phi - \cos a)\left\{g - \tfrac{1}{2}a\omega^2(\cos\phi + \cos a)\right\},$$

if α is the value of ϕ when $\dfrac{d\phi}{dt} = 0$, *i.e.* when the ring is stationary on the rod;

therefore if $g > a\omega^2$, and consequently $> \frac{1}{2} a\omega^2 (\cos\phi + \cos\alpha)$,

$$\cos\phi > \cos\alpha, \quad \phi < \alpha,$$

and the ring cannot depart from the vertical beyond the angular distance α, and may therefore perform oscillation.

59. Let M be the mass and $2a$ the length of each rod, θ their common inclination to the vertical at time t, R' the pressure of each on the horizontal plane, R their mutual horizontal action.

$$\left.\begin{aligned} \text{Then } M\frac{d^2(a\sin\theta)}{dt^2} &= R, \\ M\frac{d^2(a\cos\theta)}{dt^2} &= R' - Mg, \\ Mk^2\frac{d^2\theta}{dt} &= R'a\sin\theta - Ra\cos\theta. \end{aligned}\right\}$$

If R, R' be eliminated and the differentiations performed,

$$\tfrac{4}{3}a\frac{d^2\theta}{dt^2} = g\sin\theta,$$

$$\tfrac{2}{3}a\left(\frac{d\theta}{dt}\right)^2 = g(\cos\alpha - \cos\theta),$$

if α be the initial value of θ.

Now if $R = 0$,

$$\cos\theta\frac{d^2\theta}{dt^2} = \sin\theta\left(\frac{d\theta}{dt}\right)^2,$$

$$\cos\theta\sin\theta = 2\sin\theta(\cos\alpha - \cos\theta).$$

$$\therefore\ 3\cos\theta = 2\cos\alpha.$$

Since $\cos\alpha < \frac{3}{2}$ this equation is satisfied by a possible value of θ, and thus the rods separate before they reach the horizontal plane.

60. Let a be the length of the string when one of the bodies is stopped, x its increase of length at time t, T the tension at that time, m the mass of the moving particle;

$$\therefore\ m\frac{d^2x}{dt^2} = -T,$$

$$x = aeT;\ \text{(Hooke's law.)}$$

$$\therefore\ \frac{d^2x}{dt^2} + \frac{x}{aem} = 0,$$

$$x = A\cos\left(t\sqrt{\frac{1}{aem}} + B\right).$$

If t date from the instant of the check when V is the velocity,

$$x = V\sqrt{aem}.\sin\left(\frac{t}{\sqrt{aem}}\right),$$

and the body advances through the space $V\sqrt{aem}$ before it begins to return.

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